

# Finsler Branes and Quantum Gravity Phenomenology with Lorentz Symmetry Violations

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## Abstract

A consistent theory of quantum gravity (QG) at Planck scale almost sure contains manifestations of Lorentz local symmetry violations (LV) which may be detected at observable scales. This can be effectively described and classified by models with nonlinear dispersions and related Finsler metrics and fundamental geometric objects (nonlinear and linear connections) depending on velocity/ momentum variables. We prove that the trapping brane mechanism provides an accurate description of gravitational and matter field phenomena with LV over a wide range of distance scales and recovering in a systematic way the general relativity (GR) and local Lorentz symmetries. In contrast to the models with extra spacetime dimensions, the Einstein–Finsler type gravity theories are positively with nontrivial nonlinear connection structure, nonholonomic constraints and torsion induced by generic off–diagonal coefficients of metrics, and determined by fundamental QG and/or LV effects.

**Keywords:** quantum gravity, Lorentz violation, nonlinear dispersion, Finsler geometry, brane physics.

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## 1 Introduction and Preliminaries

There are several reasons to study generalizations of the Einstein gravity theory to models with local anisotropy, extra dimensions, analogous gravitational interactions and Finsler geometries. The first one goes in relation to

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the so-called quantum gravity (QG) phenomenology being supported by a number of ideas and research on possible observable QG effects and induced violations of Lorentz invariance (LV), see recent reviews [1, 2, 3, 4].

There were also analyzed possible production QG scenarios of mini-black holes in TeV-scale at colliders [5], or in cosmic rays [6], and Planck-scale fuzziness of spacetime [7]. For some special situations, the QG effects/objects may manifest as a non-commutative geometry [8] or on some brane-world backgrounds [9]. A series of tentative results in various approaches to QG and string theory, and computations of local quantum field theory, suggest the proposal that the Lorentz invariance may be only a low energy symmetry.

Let us consider two important arguments about nonlinear dispersion relations in QG and possible related modifications of the fundamental concepts on spacetime geometry:

1. **Nonlinear dispersions and LV in QG.** Generically, we can write for a particle of mass  $m_0$  propagating in a "slightly deformed" four dimensional (4-d) Minkowski spacetime

$$E^2 = p^2 c^2 + m_0^2 c^4 + \varphi(E, p; \mu; M_P), \quad (1)$$

where  $c$  is the light velocity,  $E$  and  $p$  are respectively the energy and momentum of the particle;  $\mu$  is some particle physics mass scale and (normally) assumes that the Planck mass  $M_P \approx 1.22 \times 10^{19} \text{GeV}$  denotes the mass scale at which the QG corrections become appreciable. The nonlinear term  $\varphi(\dots)$  encodes possible quantum matter and gravity effects and LV terms<sup>1</sup>. For  $\varphi = 0$ , we get locally the standard mass/energy/momentum relation describing a point particle in the special theory of relativity (SR). Assuming  $E \sim \frac{\partial}{\partial t}$ ,  $p_i \sim \frac{\partial}{\partial x^i}$  for some background bosonic media with "effective light velocity"  $c_s$  (see details in [1, 2, 3]), the nonlinear energy-momentum relation (1) results in

$$\omega^2 = c_s^2 k^2 + c_s^2 \left( \frac{\bar{h}}{2m_0 c_s} \right)^2 k^4 + \dots \quad (2)$$

when the "phonon" dispersion relation  $\omega \approx c_s |k|$  violates the acoustic Lorentz invariance with the wave length  $\lambda = 2\pi/|k|$ , for  $k^2 = (k_1)^2 + (k_2)^2 + (k_3)^2$  and  $\bar{h} = \text{const}$ . It is possible to derive more "sophisticate" dispersion relations with cubic on  $k$  and higher order terms

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<sup>1</sup>in explicit form, we have to consider additional dependencies and characteristic parametrizations on spacetime coordinates  $x^i$ , metric  $g_{ij}$ , spin of particle, chosen types of spacetime connections etc

and different coefficients than those in (2) if more general models of "effective media" with fermionic and/or bosonic fields are considered.

2. **Finsler generating functions from nonlinear dispersions.** Nonlinear dispersions of type (2) encode not only "energy-momentum" properties of point particles for LV. They contain also a very fundamental information about possible metric elements defining more general spacetime geometries than those postulated in SR and GR. Here, we briefly present a setup for such constructions in terms of Finsler geometry [10, 11, 12, 13]. A Minkowski metric  $\eta_{ij} = \text{diag}[-1, +1, +1, +1]$  (for  $i = 1, 2, 3, 4$ ) defines a quadratic line element in SR,

$$ds^2 = \eta_{ij} dx^i dx^j = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \quad (3)$$

with space type,  $(x^2, x^3, x^4)$ , and time like,  $x^1 = ct$ , coordinates where  $c$  is the light speed.<sup>2</sup> We can write for some classes of coordinate systems (for simplicity, omitting priming of indices and considering that indices of type  $\hat{i}, \hat{j}, \dots = 2, 3, 4$ )

$$c^2 = g_{\hat{i}\hat{j}}(x^{\hat{i}}) y^{\hat{i}} y^{\hat{j}} / \tau^2. \quad (4)$$

This formula can be used also in GR if we consider that  $g_{\hat{i}\hat{j}}(x^{\hat{i}})$  are solutions of Einstein equations. The above quadratic on  $y^{\hat{i}}$  expression can be generalized to an arbitrary nonlinear one,  $\check{F}^2(y^{\hat{j}})$ , in order to model propagation of light in anisotropic media and/or for modeling an (ether) spacetime geometry. We have to impose the condition of homogeneity,  $\check{F}(\beta y^{\hat{j}}) = \beta \check{F}(y^{\hat{j}})$  for any  $\beta > 0$ , which is necessary for description of light propagation. The formula (4) transforms into

$$c^2 = \check{F}^2(y^{\hat{j}}) / \tau^2. \quad (5)$$

Using approximations of type  $\check{F}^2(y^{\hat{j}}) \approx \left( \eta_{\hat{i}\hat{j}} y^{\hat{i}} y^{\hat{j}} \right)^r + q_{\hat{i}_1 \hat{i}_2 \dots \hat{i}_{2r}} y^{\hat{i}_1} \dots y^{\hat{i}_{2r}}$ , for  $r = 1, 2, \dots$  and  $\hat{i}_1, \hat{i}_2, \dots, \hat{i}_{2r} = 2, 3, 4$ , we can parametrize small deformations of (4) to (5). For  $r = 1$  and  $q_{\hat{i}_1 \hat{i}_2 \dots \hat{i}_{2r}} \rightarrow 0$ , we get the propagation of light rays in SR. Instead of  $\eta_{\hat{i}\hat{j}}$ , we can introduce a metric  $g_{\hat{i}\hat{j}}(x^{\hat{i}})$  from GR and include it in  $\check{F}^2$  for gravitational fields

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<sup>2</sup>Light rays can be parametrized as  $x^i(\varsigma)$  with a real smooth parameter  $0 \leq \varsigma \leq \varsigma_0$ , when  $ds^2/d\varsigma^2 = 0$ ; there is a "null" tangent vector field  $y^i(\varsigma) = dx^i/d\varsigma$ , with  $d\tau = dt/d\varsigma$ . Under general coordinate transforms  $x^{i'} = x^{i'}(x^i)$ , we have  $\eta_{ij} \rightarrow g_{i'j'}(x^k)$ ; the condition  $ds^2/d\varsigma^2 = 0$  holds always for propagation of light, i.e.  $g_{i'j'} y^{i'} y^{j'} = 0$ .

when  $\tilde{F}^2(x^i, y^{\hat{j}}) \approx \left(g_{\hat{i}\hat{j}}(x^k)y^{\hat{i}}y^{\hat{j}}\right)^r + q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}}(x^k)y^{\hat{i}_1}\ldots y^{\hat{i}_{2r}}$ . For such deformations (derived from (3) and (4)), we get generalized nonlinear homogeneous quadratic elements,

$$ds^2 = F^2(x^i, y^j) \approx -(cdt)^2 + g_{\hat{i}\hat{j}}(x^k)y^{\hat{i}}y^{\hat{j}}\left[1 + \frac{1}{r}\frac{q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}}(x^k)y^{\hat{i}_1}\ldots y^{\hat{i}_{2r}}}{\left(g_{\hat{i}\hat{j}}(x^k)y^{\hat{i}}y^{\hat{j}}\right)^r}\right] + O(q^2), \quad (6)$$

when  $F(x^i, \beta y^j) = \beta F(x^i, y^j)$ , for any  $\beta > 0$ . A value  $F$  is called a fundamental (generating) Finsler function usually satisfying the condition that the Hessian

$$^F g_{ij}(x^i, y^j) = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j} \quad (7)$$

is not degenerate, see details in [14, 15, 16, 17]. For light rays, the nonlinear element (6) defines a nonlinear dispersion relation between the frequency  $\omega$  and the wave vector  $k_i$ ,<sup>3</sup>

$$\omega^2 = c^2 [g_{\hat{i}\hat{j}} k^{\hat{i}} k^{\hat{j}}]^2 \left(1 - \frac{1}{r} \frac{q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}} k^{\hat{i}_1} \ldots k^{\hat{i}_{2r}}}{[g_{\hat{i}\hat{j}} k^{\hat{i}} k^{\hat{j}}]^{2r}}\right). \quad (8)$$

The dispersion relations should be parametrized and computed differently for various classes of theories formulated in terms of Finsler geometry and generalizations. Here we cite a series of works on very special relativity [18, 19], generalized (super) Finsler gravity and LV induced from string gravity [20, 21, 22], double special relativity [23, 24], Finsler–Higgs mechanism [25], Finsler black holes/ellipsoids induced by noncommutative variables [26]. In particular, we can chose such subsets of coefficients  $q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}}$  when (8) transforms into (2).

The main conclusion we derive from above points 1 and 2 is that various classical and quantum gravity theories are with local nonlinear dispersions of type (2) and/or (8). Such theories are positively with LV and can be characterized geometrically by nonlinear Finsler type quadratic elements (6) constructed as certain deformations of standard quadratic elements for Minkowski (3) and/or pseudo–Riemannian spacetimes. This results in geometric constructions on tangent,  $TV$  (with local coordinated  $u^\alpha = (x^i, y^a)$ , where  $y^a$  label fiber coordinates; we shall write in brief  $u = (x, y)$ ). We can elaborate physical models on cotangent,  $T^*V$  (with local coordinates

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<sup>3</sup>for simplicity, we can consider such a relation in a fixed point  $x^k = x^k_{(0)}$ , when  $g_{\hat{i}\hat{j}}(x^k_0) = g_{\hat{i}\hat{j}}$  and  $q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}} = q_{\hat{i}_1\hat{i}_2\ldots\hat{i}_{2r}}(x^k_0)$

$\tilde{u}^\alpha = (x^i, p_a)$ , where  $p_a$  label co-fiber coordinates), bundles to a curved spacetime manifold  $V$  (with local coordinates  $x^i = (x^1, x^2, x^3, x^4)$  of pseudo-Euclidean signature). Constructions on  $TM$  and  $T^*V$  are typical for Finsler-Lagrange, and/or Cartan-Hamilton geometries, and generalizations, see details and references in [14, 15, 17, 13]. In modern particle physics and cosmology, see [27, 28, 29], there is a renewed interest in Finsler geometry applications, see reviews of results and critical remarks in [30, 31, 32, 33, 34].

Nonlinear dispersions and associated Finsler like generating functions suggest the idea that a self-consistent QG theory may be constructed not just for a 4-d pseudo-Riemannian spacetime  $V$  but for certain Finsler type extensions on  $TV$  and/or  $T^*V$ . Following a nonholonomic generalization of Fedosov deformation quantization, such quantum gravity models were studied in [33, 34]. Roughly speaking, a QG model with some generalized nonlinear dispersions, and associated fundamental Finsler structures, should replace GR at very short distances approaching the Planck length,  $l_P \simeq \sqrt{\frac{4G\hbar}{c^3}} \simeq 1.6 \times 10^{-33} \text{ cm}$ , where  ${}^4G$  is the 4-d Newton constant and  $\hbar = h/2\pi$  is the Planck constant.

Over short distances, we have certain modifications of GR which seem to be of Finsler type with additional depending "velocity/momentum" type coordinates. A Finsler spacetime geometry/ gravity model is not completely determined only by its nonlinear quadratic element  $F(x, y)$  (6), or Hessian (7). It is completely stated after we choose (following certain physical arguments) what types of metric tensor,  ${}^F\mathbf{g}$ , nonlinear connection (N-connection),  ${}^F\mathbf{N}$ , and linear connection,  ${}^F\mathbf{D}$ , are canonically induced by a generating Finsler function  $F(x, y)$  on  $\widetilde{TV} \equiv TV/\{0\}$  (we exclude the null sections  $\{0\}$  over  $TM$ ).<sup>4</sup> The nature of QG and LV effects derived in certain theoretical construction is related to a series of assumptions on fundamental spacetime structure and considered classes of fundamental equations, conservation laws and symmetries. For instance, it depends on the fact if  ${}^F\mathbf{D}$  and  ${}^F\mathbf{g}$  are compatible, or not; what type of torsion  ${}^F\mathcal{T}$  of  ${}^F\mathbf{D}$  is induced by  $F$  and/or  ${}^F\mathbf{g}$  and  ${}^F\mathbf{N}$ , if there are considered compact and/or non-compact extra/velocity/momentum type dimensions etc (in section 2, we present rigorous definitions of such geometric/physical objects).

In this work, our focus is on LV effects and QG phenomenology de-

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<sup>4</sup>Following our conventions [30, 31, 17], we use "boldface" symbols for spaces/ geometrical/physical objects endowed with /adapted to nonlinear connection structure, see definitions in next section; we also put left up/low labels in order to emphasize that a geometric/physical object is completely defined/induced by a corresponding fundamental generating function, for instance, that  ${}^F\mathbf{g}$  is completely and uniquely determined by  $F$ .

terminated by mechanisms for trapping/locallyzing gravitational and matter fields from a Finsler spacetime on  $TV$  to a 4-d observable pseudo-Riemannian spacetime  $V$ . Such ideas were originally considered in brane gravity, see [35, 36, 37] and [38, 39], with a non-compact extra dimension coordinate. In Finsler gravity theories on tangent bundles of 3-d and/or 4-d pseudo-Riemannian spacetimes, there are considered respectively 3+3 and/or 4+4 dimensional locally anisotropic spacetime models determined by data  $[F : {}^F\mathbf{g}, {}^F\mathbf{N}, {}^F\mathbf{D}]$ . This contains a more rich geometric structure than that for Einstein spaces determined by a metric  $\mathbf{g}$ , and its unique torsionless and metric compatible Levi-Civita connection  ${}^g\nabla$ .

Warped Finsler like configurations and related trapping "isotropization" have to be adapted to the N-connection structure (as we studied for the case of locally anisotropic black holes and propagating solitonically black holes and wormholes [40]). We can consider solutions with both types of exponential and non-exponential factors by introducing non-gravitational interactions or considering a pure gravitational trapping mechanism for all types of spin fields similarly to 5-d and 6-d pseudo-Riemannian configurations in [41, 42, 43, 44]. The physics of locally isotropic brane theories with extra dimensions and the Finsler-brane models are very different even the LV effects (see [45, 32, 46] for locally isotropic branes in 5-d) can be computed in both cases<sup>5</sup>.

The portion of this paper developing conceptual and theoretical issues of Finsler gravity and brane theories spans sections 2 and 3. It begins in section 2 with a review of the Einstein-Finsler gravity model and the anholonomic deformation method of constructing exact solutions for gravitational field equations. Section 3 concerns explicit Finsler-brane solutions in 3+3 and 4+4 dimensional gravity on tangent bundles. Finally, in section 4 we conclude the results. For convenience, in Appendices we provide two important Theorems and relevant computations on constructing exact solutions in Einstein and Finsler gravity theories. Throughout the paper, we follow the conventions of Refs. [31, 17] and [13] where possible.

## 2 Einstein-Finsler Gravity

In general, there are two different classes of Finsler gravity models which can be constructed on a  $\mathbf{TV} = (TV, \pi, V)$ , where  $TV$  is the total space,  $\pi$

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<sup>5</sup>the Finsler configurations are of different nature when the generating functions are determined by certain general coefficients in a QG model, and related LV, which is not the case, for instance, of Randall-Sundrum branes

is a surjective projection and  $V, \dim V = n$ . In this work, we consider that  $V$  is a pseudo-Riemannian/Einstein manifold of dimension  $n = 2, 3$ , or  $4$ ), is the base manifold, see details and critical remarks in Refs [30, 13, 31, 17].

The first class of theories is with metric noncompatible linear Finsler connections  ${}^F\mathbf{D}$  when  ${}^F\mathbf{D} {}^F\mathbf{g} = {}^F\mathbf{Q} \neq \mathbf{0}$  (the typical example is that when  ${}^F\mathbf{D} = {}^{Ch}\mathbf{D}$  is the Chern connection for which the total  ${}^{Ch}\mathbf{Q}$  is not zero but torsion vanishes,  ${}^{Ch}\mathcal{T} = 0$ ). Because of nonmetricity, it seems that there are a number of conceptual/theoretical and technical problems with definition of spinors and Dirac operators, conservations laws and performing quantization of such theories [30, 13, 31]. In our opinion, such geometries have less perspectives for applications in standard particle physics and "simple" modifications, for instance, for purposes of modern cosmology.

The second class of Finsler gravity models is with such  ${}^F\mathbf{D}$  which are metric compatible, i.e.  ${}^F\mathbf{D} {}^F\mathbf{g} = 0$ . Such a locally anisotropic gravity theory is positively with nontrivial torsion,  ${}^F\mathcal{T} \neq 0$ . A very important property is that there are  ${}^F\mathbf{D}$  when  ${}^F\mathcal{T}$  is completely defined by the total Finsler metric structure  ${}^F\mathbf{g}$  and a prescribed nonlinear connection (N-connection)  ${}^F\mathbf{N}$ . For instance, this is the case of canonical distinguished connection (d-connection)  ${}^F\mathbf{D} = \hat{\mathbf{D}}$ , see details in [15, 31, 17], and the Cartan d-connection  ${}^F\mathbf{D} = \tilde{\mathbf{D}}$ , see formula (20) below. There are preferred constructions with  $\hat{\mathbf{D}}$  because it defines canonically an almost Kähler structure (for instance, this is important for deformation/ A-brane quantization of gravity [33, 34]).

An **Einstein-Finsler gravity** theory (EFG), we consider a model of gravity on  $TV$  defined by data  $[F : {}^F\mathbf{g}, {}^F\mathbf{N}, {}^F\mathbf{D} = \tilde{\mathbf{D}}]$  and corresponding gravitational field equations in such variables (see section 2.2) following the same principles (postulates) as in GR stated by data  $[\mathbf{g}, {}^g\nabla]$ . Additionally, we suppose that there is a trapping/warped mechanism defined by explicit solutions of (Finsler type) gravitational field equations which in classical limits for  $l_P \rightarrow 0$ , when  $\text{EFG} \rightarrow \text{GR}$ , determining QG corrections to gravitational and matter field interactions at different scales depending on the class of considered models and solutions.

## 2.1 Fundamental objects in EFG

A (pseudo) Finsler space  $F^n = (V, F)$  corresponding, for instance, to a (pseudo) Riemannian manifold  $V$  of signature  $(-, +, +, \dots)$  consists of a Finsler metric (fundamental/generating function)  $F(x, y)$  (6) defined as a real valued function  $F : TV \rightarrow \mathbb{R}$  with the properties that the restriction of

$F$  to  $\widetilde{TM}$  is a function 1) positive; 2) of class  $C^\infty$  and  $F$  is only continuous on  $\{0\}$ ; 3) positively homogeneous of degree 1 with respect to  $y^i$ , i.e.  $F(x, \beta y) = |\beta|F(x, y)$ ,  $\beta \in \mathbb{R}$ ; and 4) the Hessian  ${}^F g_{ij} = (1/2)\partial^2 F / \partial y^i \partial y^j$  (7) defined on  $\widetilde{TV}$ , is nondegenerate (for Finsler spaces, this condition is changed into that  ${}^F g_{ij}$  is positive definite. In brief, a Finsler space is a Lagrange space with effective Lagrangian  $L = F^2$ .<sup>6</sup>

### 2.1.1 The canonical (Finsler) N-connection

One of three fundamental geometric objects induced by a Finsler metric  $F$ , and defining a Finsler space, is the nonlinear connection (N-connection). A N-connection  $\mathbf{N}$  is by definition ( $:=$ ) a Whitney sum

$$TTV := hTV \oplus vTV. \quad (9)$$

Geometrically, this an example of nonholonomic (equivalentl, anholonomic, or non-integrable) distribution with conventional horizontal (h) – vertical (v) decomposition/ splitting which can be considered for the module of vector fields  $\chi(TTV)$  on  $TV$ . For instance,  $\mathbf{Y} = {}^h\mathbf{Y} + {}^v\mathbf{Y}$  for any vector  $\mathbf{Y} \in \chi(TTV)$ , where  ${}^h\mathbf{Y} \doteq h\mathbf{Y} \in \chi(hTV)$  and  ${}^v\mathbf{Y} \doteq v\mathbf{Y} \in \chi(vTV)$ .

There is a canonical N-connection structure  $\mathbf{N} = {}^c\mathbf{N}$  which is defined by  $F$  following such arguments. Considering that  $L = F^2$  is a regular Lagrangian (i.e. with nondegenerate  ${}^F g_{ij}$  (7)) and define the action integral  $S(\tau) = \int_0^1 L(x(\tau), y(\tau))d\tau$ , with  $y^k(\tau) = dx^k(\tau)/d\tau$ , for  $x(\tau)$  parametrizing smooth curves on  $V$  with  $\tau \in [0, 1]$ . By straightforward computations, we can prove that the Euler–Lagrange equations of  $S(\tau)$ , i.e.  $\frac{d}{d\tau} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0$ , are equivalent to the "nonlinear geodesic" (equivalently, semi-spray) equations  $\frac{d^2 x^k}{d\tau^2} + 2G^k(x, y) = 0$ , where  $G^k = \frac{1}{4}g^{kj} \left( y^i \frac{\partial^2 L}{\partial y^j \partial x^i} - \frac{\partial L}{\partial x^j} \right)$  defines the canonical N-connection  ${}^c\mathbf{N} = \{ {}^cN_j^a \}$ , where  ${}^cN_j^a = \frac{\partial G^a(x, y)}{\partial y^j}$ .

Under general (co) frame/coordinate transform,  $\mathbf{e}^\alpha \rightarrow \mathbf{e}^{\alpha'} = e^{\alpha'}_\alpha \mathbf{e}^\alpha$  and/or  $u^\alpha \rightarrow u^{\alpha'} = u^{\alpha'}(u^\alpha)$ , preserving the splitting (9), we transform  ${}^cN_j^a \rightarrow N_{j'}^{a'}$ , when  $\mathbf{N} = N_{i'}^{a'}(u)dx^{i'} \otimes \frac{\partial}{\partial y^{a'}}$  is given locally by a set of coefficients  $\{N_{j'}^{a'}\}$ . Hereafter, we shall omit priming, underlying etc of indices if that will not result in ambiguities.

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<sup>6</sup>Similar theories can be elaborated for (pseudo) Lagrange spaces and generalizations as it is provided in [15, 31, 17]. This way, we can construct different "analogous gravity" and geometric mechanics models. Here we also note that Finsler–Lagrange variables can be introduced even in Einstein gravity which is very convenient for constructing exact solutions and developing certain models of QG.



### 2.1.2 Sasaki types lifts of metrics in Finsler spaces

For a fundamental Finsler function  $F(x, y)$ , we can construct a canonical (Sasaki type) metric structure

$${}^F\mathbf{g} = {}^Fg_{ij}(x, y) e^i \otimes e^j + (l_P)^2 {}^Fg_{ij}(x, y) {}^F\mathbf{e}^i \otimes {}^F\mathbf{e}^j, \quad (10)$$

$$e^i = dx^i \text{ and } {}^F\mathbf{e}^a = dy^a + {}^FN_i^a(u)dx^i, \quad (11)$$

where  ${}^F\mathbf{e}^\mu = (e^i, {}^F\mathbf{e}^a)$  (11) is the dual to  ${}^F\mathbf{e}_\alpha = ({}^F\mathbf{e}_i, e_a)$ , for

$${}^F\mathbf{e}_i = \frac{\partial}{\partial x^i} - {}^FN_i^a(u) \frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}. \quad (12)$$

We shall put the square of an effective Planck length  $l_P$  before the v-part of metric (10) if we shall want to have the same dimensions for the h- and v-components of metric when coordinates have the dimensions  $[x^i] = cm$  and  $[y^i \sim dx^i/ds] = cm/cm$ .

Using frame transforms  $e^{\alpha'} = e^{\alpha'}_\alpha e^\alpha$ , any metric

$$\mathbf{g} = g_{\alpha\beta} du^\alpha \otimes du^\beta \quad (13)$$

on  $TM$ ,<sup>7</sup> including  ${}^F\mathbf{g}$  (10), can be represented in N-adapted form

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + (l_P)^2 h_{ab}(x, y) \mathbf{e}^a \otimes \mathbf{e}^b, \quad (14)$$

for an N-adapted base  $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$ , where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \quad (15)$$

and the dual frame (coframe) structure is  $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$ , for

$$e^i = dx^i \text{ and } \mathbf{e}^a = dy^a + N_i^a(u)dx^i. \quad (16)$$

The local bases induced by N-connection structure, for instance, (15) satisfy nontrivial nonholonomy relations of type

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad (17)$$

with (antisymmetric) nontrivial anholonomy coefficients  $W_{ia}^b = \partial_a N_i^b$  and  $W_{ji}^a = \Omega_{ij}^a$  determined by the coefficients of curvature of N-connection.

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<sup>7</sup>a dual local coordinate basis as  $du^\beta = (dx^j, dy^b)$ , when  $\partial_\alpha = \partial/\partial u^\alpha = (\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a)$

The above formulas define h- and v-splitting of metrics on  $TM$ , respectively,  ${}^h\mathbf{g} = \{g_{ij}(u)\}$  and  ${}^v\mathbf{g} = \{h_{ab}(u)\}$ . Extending the principle of general covariance from  $V$  to  $TV$ , i.e. from GR to EFG, we can work equivalently with any parametrization of metrics in the form (10), (13), or (14). The first parametrization show in explicit form that our gravity model is for a Finsler spacetime, the second one states the coefficients of metric with respect to local coordinate (co) bases and the third one will be convenient for constructing exact solutions in EFG.

### 2.1.3 Canonical linear/distinguished connections

For any Finsler metric  ${}^F\mathbf{g}$  (10), we can compute in standard form the Levi-Civita connection  ${}^F\nabla$ . But such a linear connection is not used in Finsler geometry because it is not adapted to the N-connection structure. We have to revise the concept of linear connection for nonholonomic bundles/manifolds enabled with splitting of type (9): A distinguished connection (d-connection) is a linear connection  $\mathbf{D}$  preserving by parallelism the N-connection splitting (9).<sup>8</sup>

To a d-connection  $\mathbf{D} = ({}^hD, {}^vD) = (L^i_{jk}, C^i_{jc})$ , for  $L^i_{jk} = L^a_{bk}$  and  $C^i_{jc} = C^a_{bc}$  (with a chosen contraction for h- and v-indices), we can associate a 1-form  $\Gamma^\alpha_\beta = [\Gamma^i_j, \Gamma^a_b]$  with

$$\Gamma^i_j = \Gamma^i_{j\gamma} \mathbf{e}^\gamma = L^i_{jk} e^k + C^i_{jc} \mathbf{e}^c, \quad \Gamma^a_b = \Gamma^a_{b\gamma} \mathbf{e}^\gamma = L^a_{bk} e^k + C^a_{bc} \mathbf{e}^c.$$

The torsion,  $\mathcal{T} = \{\mathbf{T}^\alpha_{\beta\gamma}\}$ , and curvature,  $\mathcal{R} = \{\mathbf{R}^\alpha_{\beta\gamma\tau}\}$ , tensors of a d-connection  $\mathbf{D}$  are defined and computed in usual forms as for linear connections for any  $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \chi(TTV)$ . Using Cartan's structure equations

$$\begin{aligned} de^i - e^k \wedge \Gamma^i_k &= -\mathcal{T}^i, \quad d\mathbf{e}^a - \mathbf{e}^b \wedge \Gamma^a_b = -\mathcal{T}^a, \\ d\Gamma^i_j - \Gamma^k_j \wedge \Gamma^i_k &= -\mathcal{R}^i_j, \end{aligned} \quad (18)$$

we can compute the N-adapted coefficients of torsion and curvature, see details in [15, 31, 17]. For instance, an explicit computation results in

$$\mathcal{T}^i = C^i_{jc} e^i \wedge \mathbf{e}^c \text{ and } \mathcal{T}^a = -\frac{1}{2} \Omega^a_{ij} e^i \wedge e^j + (e_b N^a_i - L^a_{bi}) e^i \wedge \mathbf{e}^b, \quad (19)$$

with nontrivial values (anti-symmetric on lower indices) of  $T^i_{jc} = -T^i_{cj} = C^i_{jc}$ ,  $T^a_{ji} = -T^a_{ij} = \frac{1}{2} \Omega^a_{ij}$ ,  $T^a_{bi} = -T^a_{ib} = e_b N^a_i - L^a_{bi}$ .

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<sup>8</sup>In Lagrange-Finsler geometry, there are used the terms distinguished tensor/ vector / spinor connection etc (d-tensor, d-vector, d-spinor, d-connection etc) [15, 31, 17] for the corresponding geometric objects defined with respect to N-adapted (co) bases.

For a metric structure  $\mathbf{g} = [g_{ij}, h_{ab}]$  (14), there is a unique normal d-connection  $\tilde{\mathbf{D}}$  which is metric compatible,  $\tilde{\mathbf{D}} \mathbf{g} = \mathbf{0}$ , and with vanishing  $hhh$ - and  $vvv$ -components ( ${}^h\tilde{\mathcal{T}}(h\mathbf{X}, h\mathbf{Y}) = 0$  and  ${}^v\tilde{\mathcal{T}}(v\mathbf{X}, v\mathbf{Y}) = 0$ , for any vectors  $\mathbf{X}$  and  $\mathbf{Y}$ ) of torsion  $\tilde{\mathcal{T}}$  computed following formulas (19). If  $\mathbf{g} = {}^F\mathbf{g}$ , we get the coefficients of the so-called Cartan d-connection in Finsler geometry [14, 15, 31, 17]. We can verify that locally the normal d-connection  $\tilde{\mathbf{D}} = ({}^h\tilde{\mathbf{D}}, {}^v\tilde{\mathbf{D}})$  is given respectively by coefficients  $\tilde{\Gamma}^\alpha_{\beta\gamma} = (\tilde{L}^a_{bk}, \tilde{C}^a_{bc})$ ,

$$\tilde{L}^i_{jk} = \frac{1}{2}g^{ih}(\mathbf{e}_k g_{jh} + \mathbf{e}_j g_{kh} - \mathbf{e}_h g_{jk}), \tilde{C}^a_{bc} = \frac{1}{2}h^{ae}(e_b h_{ec} + e_c h_{eb} - e_e h_{bc}), \quad (20)$$

are computed with respect to N-adapted frames. The covariant h-derivative is  ${}^h\tilde{\mathbf{D}} = \{\tilde{L}^i_{jk}\}$  and v-derivative is  ${}^v\tilde{\mathbf{D}} = \{\tilde{C}^a_{bc}\}$ . The torsion coefficients  $\tilde{\mathbf{T}}^\alpha_{\beta\gamma}$  of  $\tilde{\mathbf{D}}$  are  $\tilde{T}^i_{jk} = 0$  and  $\tilde{T}^a_{bc} = 0$  but with non-zero cross coefficients,  $\tilde{T}^a_{ij} = \Omega^a_{ij}, \tilde{T}^a_{ib} = e_b N^a_i - \tilde{L}^a_{bi}$ .

#### 2.1.4 Finsler variables in (pseudo) Riemannian geometry

Finsler variables can be introduced not only on  $TM$  but also, via corresponding nonholonomic distributions, on any pseudo-Riemannian manifold [45, 32, 34, 31]  $\mathbf{V}, \dim \mathbf{V} = 2n, n \geq 2$ , enabled with metric structure  $\mathbf{g}$ . On such a manifold, we can prescribe any type of nonholonomic frames/ distributions. For instance, we can choose a distribution defined by a regular generating function of necessary type homogeneity,  $F(x, y)$ , when coordinates  $u = (x, y)$  are local ones on  $\mathbf{V}$ , with nondegenerate Hessian  ${}^Fg_{ij}$ , and define  $\mathbf{g} = {}^F\mathbf{g}$ . We model on  $\mathbf{V}$  a Finsler geometry if we construct from  ${}^F\mathbf{g}$ , in a unique form, the Cartan d-connection  $\tilde{\mathbf{D}}$ .

In "standard" variables, a (pseudo) Riemannian geometry is characterized by the Levi-Civita connection  $\nabla$ .<sup>9</sup> We have

$$\tilde{\mathbf{D}} = {}^F\nabla + \tilde{\mathbf{Z}}, \quad (21)$$

where the distortion tensor  ${}^F\tilde{\mathbf{Z}}$  is determined by the torsion  $\tilde{\mathcal{T}}$ , see explicit coefficients (19). All such geometric objects (i.e.  $\tilde{\mathbf{D}}, {}^F\nabla, \tilde{\mathbf{Z}}$ ) are completely defined by the same metric structure  $\mathbf{g}$ . Any geometric (pseudo) Riemannian data  $(\mathbf{g}, \nabla)$  can be transformed equivalently into  $(\mathbf{g} = {}^F\mathbf{g}, \tilde{\mathbf{D}})$  and inversely.

The question of (at least formal) equivalence of two gravity theories given by data/ variables  $[F : {}^F\mathbf{g}, {}^F\mathbf{N}, {}^F\mathbf{D} = \tilde{\mathbf{D}}]$  or  $[\mathbf{g} = {}^F\mathbf{g}, {}^g\nabla = {}^F\nabla = \tilde{\mathbf{D}} - \tilde{\mathbf{Z}}]$  (on  $TV$ , or  $\mathbf{V}$ ) depends on the type of gravitational field equations (for  $\tilde{\mathbf{D}}$  or  $\nabla$ ) and matter field sources are postulated for a model of relativity theory.

<sup>9</sup>By definition, it is metric compatible,  $\nabla \mathbf{g} = 0$ , and torsionless,  ${}^\nabla\mathcal{T} = 0$ .

## 2.2 Field equations in EFG

We can elaborate a Finsler gravity theory on  $TM$  using the d-connection  $\tilde{\mathbf{D}}$  and following in general lines the same postulates as in GR. Such a model present a minimal metric compatible Finsler extension of the Einstein gravity but for the generating function  $F$ .

The curvature 2-form of  $\tilde{\mathbf{D}} = \{\tilde{\Gamma}_{\beta\gamma}^\alpha\}$  is computed (see (18))

$$\tilde{\mathcal{R}}_\gamma^\tau = \tilde{\mathbf{R}}_{\gamma\alpha\beta}^\tau \mathbf{e}^\alpha \wedge \mathbf{e}^\beta = \frac{1}{2} \tilde{R}_{jkh}^i e^k \wedge e^h + \tilde{P}_{jka}^i e^k \wedge e^a + \frac{1}{2} \tilde{S}_{jcd}^i e^c \wedge e^d,$$

when the nontrivial N-adapted coefficients of curvature  $\tilde{\mathbf{R}}_{\beta\gamma\tau}^\alpha$  are

$$\begin{aligned} \tilde{R}_{hjk}^i &= \mathbf{e}_k \tilde{L}_{hj}^i - \mathbf{e}_j \tilde{L}_{hk}^i + \tilde{L}_{hj}^m \tilde{L}_{mk}^i - \tilde{L}_{hk}^m \tilde{L}_{mj}^i - \tilde{C}_{ha}^i \Omega_{kj}^a, \\ \tilde{P}_{jka}^i &= e_a \tilde{L}_{jk}^i - \tilde{\mathbf{D}}_k \tilde{C}_{ja}^i, \quad \tilde{S}_{bcd}^a = e_d \tilde{C}_{bc}^a - e_c \tilde{C}_{bd}^a + \tilde{C}_{bc}^e \tilde{C}_{ed}^a - \tilde{C}_{bd}^e \tilde{C}_{ec}^a. \end{aligned}$$

The Ricci tensor  $\tilde{Ric} = \{\tilde{\mathbf{R}}_{\alpha\beta}\}$  is defined by contracting respectively the components of curvature tensor,  $\tilde{\mathbf{R}}_{\alpha\beta} \doteq \tilde{\mathbf{R}}_{\alpha\beta\tau}^\tau$ . The scalar curvature is  ${}^s\tilde{\mathbf{R}} \doteq \mathbf{g}^{\alpha\beta} \tilde{\mathbf{R}}_{\alpha\beta} = g^{ij} \tilde{R}_{ij} + h^{ab} \tilde{R}_{ab}$ , where  $\tilde{R} = g^{ij} \tilde{R}_{ij}$  and  $\tilde{S} = h^{ab} \tilde{R}_{ab}$  are respectively the h- and v-components of scalar curvature.

The gravitational field equations for our Finsler gravity model with metric compatible d-connection  ${}^F\mathbf{D} = \tilde{\mathbf{D}}$ ,

$$\tilde{\mathbf{E}}_{\beta\delta} = \tilde{\mathbf{R}}_{\beta\delta} - \frac{1}{2} \mathbf{g}_{\beta\delta} {}^s\tilde{R} = \tilde{\Upsilon}_{\beta\delta} \quad (22)$$

can be introduced in geometric and/or variational forms on  $TM$ , similarly to Einstein equations in GR,

$${}_1E_{\beta\delta} = {}_1R_{\beta\delta} - \frac{1}{2} \mathbf{g}_{\beta\delta} {}^sR = {}_1\Upsilon_{\beta\delta}, \quad (23)$$

where all values (the Einstein and Ricci tensors, respectively,  ${}_1E_{\beta\delta}$  and  ${}_1R_{\beta\delta}$ , scalar curvature,  ${}^sR$ , and the energy-momentum tensor,  ${}_1\Upsilon_{\beta\delta}$ ) are for the Levi-Civita connection  ${}^F\nabla$  computed for the same  $\mathbf{g}_{\beta\delta} = {}^F\mathbf{g}_{\beta\delta}$ .

A source  $\tilde{\Upsilon}_{\beta\delta}$  can be defined following certain geometric and/or N-adapted variational principles for matter fields, see such examples in [17]. An important property of the equations (22) is that it can be integrated in very general forms. On exact solutions for such equations (related to black hole physics, locally anisotropic thermodynamics etc) see [45, 32, 34, 31, 17] and references therein. Finsler modified Einstein equations of type (22) can

be such way constructed that they would be equivalent to the Einstein equations for  $\nabla$ .<sup>10</sup> Such an equivalence is important if we reformulate the GR theory in Finsler, or almost Kähler variables [45, 32, 34], but there are not strong theoretical and/or experimental arguments to impose such conditions for Finsler gravity theories on  $TM$ .

Finally, we emphasize that the EFG theory is positively with nontrivial torsion structure  $\tilde{\mathbf{T}}^\alpha_{\beta\gamma}$  induced by fundamental generating function  $F(x, y)$ . This torsion is completely defined by certain off-diagonal coefficients of the metric structure  ${}^F\mathbf{g}$ , including  ${}^F\mathbf{N}$ .

### 3 Finsler–Branes

Examples of Einstein–Finsler gravity model and QG phenomenology can be elaborated for metrics  ${}^F\mathbf{g}$ , see parametrizations (10) and (14), transforming into Einstein metrics for  $l_P \rightarrow 0$ . In the classical limit, the gravitational physics is satisfactory described by GR (perhaps with certain exceptions related to accelerating Universes and dark energy/matter problems (see [27, 17, 28, 29, 30])). In this section, we study scenarios of QG phenomenology and LV when classical 4-d Einstein spacetimes are embedded into 8-d Finsler spaces with non-factorizable velocity type coordinates. Experimentally, the light velocity is finite and metrics in GR do not depend explicitly on velocity/momentum type variables which can be modelled via trapping/warping solutions in EFG.

#### 3.1 General ansatz and integrable field equations

The system of equations (22) can be integrated in very general forms (following geometric methods reviewed in details in Refs. [45, 32]). In this paper, we can use a simplified approach because our 8-d Finsler gravity models are with Killing symmetries and Finsler branes can be described by some off-diagonal ansatz for metrics and connections.<sup>11</sup>

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<sup>10</sup>This is possible for the normal/ Cartan d-connection  $\tilde{\mathbf{D}}$  being completely defined by  $\mathbf{g}_{\beta\delta}$  (14) and if  $\tilde{\mathbf{\Upsilon}}_{\beta\delta} = {}^{matter}\mathbf{\Upsilon}_{\beta\delta} + {}^z\mathbf{\Upsilon}_{\beta\delta}$  are derived in such a way that they contain contributions from 1) the N-adapted energy-momentum tensor (defined variationally following the same principles as in GR but on  $TV$ ) and 2) the distortion of the Einstein tensor in terms of  $\tilde{\mathbf{Z}}$  (21),  $\tilde{\mathbf{Z}}_{\beta\delta} = {}^E\mathbf{Z}_{\beta\delta} + {}^z\mathbf{Z}_{\beta\delta}$ , for  ${}^z\mathbf{Z}_{\beta\delta} = {}^z\mathbf{\Upsilon}_{\beta\delta}$ . The value  ${}^z\mathbf{Z}_{\beta\delta}$  is computed by introducing  $\tilde{\mathbf{D}} = {}^F\nabla + \tilde{\mathbf{Z}}$  into (22) and corresponding contractions of indices in order to find the Ricci d-tensor and scalar curvature.

<sup>11</sup>For convenience, we provide in Appendix two theorems on constructing exact solutions for a 4-d Einstein–Finsler toy model which is exactly integrable. Various extensions of the

It possible to extend GR theory to holonomic 8-d models on tangent bundle considering a trivial N-connection/Finsler structure for the EFG when solutions with diagonal metrics play an important role. To select a more realistic model of velocity/momentum depending gravity, we have to solve the 8-d Einstein equations (23) (defining a "velocity" depending type of scalar-tensor gravity theory, see discussion in Ref. [13]) and compare such classes of solutions with generic off-diagonal ones and nontrivial d-torsion and N-connection structures constructed for Finsler gravity.

We use an ansatz which via frame transform can be parametrized

$$\begin{aligned}
\mathbf{g} = & \phi^2(y^5)[g_1(x^k) e^1 \otimes e^1 + g_2(x^k) e^2 \otimes e^2 + h_3(x^k, v) \mathbf{e}^3 \otimes \mathbf{e}^3 + \\
& h_4(x^k, v) \mathbf{e}^4 \otimes \mathbf{e}^4] + (l_P)^2 [h_5(x^k, v, y^5) \mathbf{e}^5 \otimes \mathbf{e}^5 + h_6(x^k, v, y^5) \mathbf{e}^6 \otimes \mathbf{e}^6] \\
& + (l_P)^2 [h_7(x^k, v, y^5, y^7) \mathbf{e}^7 \otimes \mathbf{e}^7 + h_8(x^k, v, y^5, y^7) \mathbf{e}^8 \otimes \mathbf{e}^8], \quad (24) \\
\mathbf{e}^3 = & dv + w_i dx^i, \quad \mathbf{e}^4 = dy^4 + n_i dx^i, \quad \mathbf{e}^5 = dy^5 + {}^1w_i dx^i + {}^1w_3 dv + {}^1w_4 dy^4, \\
\mathbf{e}^6 = & dy^6 + {}^1n_i dx^i + {}^1n_3 dv + {}^1n_4 dy^4, \\
\mathbf{e}^7 = & dy^7 + {}^2w_i dx^i + {}^2w_3 dv + {}^2w_4 dy^4 + {}^2w_5 dy^5 + {}^2w_6 dy^6, \\
\mathbf{e}^8 = & dy^8 + {}^2n_i dx^i + {}^2n_3 dv + {}^2n_4 dy^4 + {}^2n_5 dy^5 + {}^2n_6 dy^6,
\end{aligned}$$

for nontrivial N-connection coefficients

$$\begin{aligned}
N_i^3 &= w_i(x^k, v), N_i^4 = n_i(x^k, v); \quad (25) \\
N_i^5 &= {}^1w_i(x^k, v, y^5), N_3^5 = {}^1w_3(x^k, v, y^5), N_4^5 = {}^1w_4(x^k, v, y^5); \\
N_i^6 &= {}^1n_i(x^k, v, y^5); N_3^6 = {}^1n_3(x^k, v, y^5), N_4^6 = {}^1n_4(x^k, v, y^5); \\
N_i^7 &= {}^2w_i(x^k, v, y^7), N_3^7 = {}^2w_3(x^k, v, y^7), N_4^7 = {}^2w_4(x^k, v, y^7), \\
& N_5^7 = {}^2w_5(x^k, v, y^7), N_6^7 = {}^2w_6(x^k, v, y^7); \\
N_i^8 &= {}^2n_i(x^k, v, y^7), N_3^8 = {}^2n_3(x^k, v, y^7), N_4^8 = {}^2n_4(x^k, v, y^7), \\
& N_5^8 = {}^2n_5(x^k, v, y^7), N_6^8 = {}^2n_6(x^k, v, y^7).
\end{aligned}$$

The local coordinates in the above ansatz (24) are labeled in the form  $x^i = (x^1, x^2)$ , for  $i, j, \dots = 1, 2$ ;  $y^3 = v$ .

Our goal is to construct and analyze physical implications of solutions of equations (23) and (22) defined by ansatz (24) with, respectively, trivial and non-trivial N-connection coefficients (25).

### 3.2 Holonomic brane configurations

A trapping scenario with diagonal metric from QG with LV to GR can be constructed for an ansatz of type (24) with zero N-connection coefficients

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outlined there anholonomic deformation method to 6-d and 8-d Finsler brane spacetimes with nontrivial N-connection structures are straightforward.

(25) when  $h_5, h_7, h_8 = \text{const}$  and data  $[g_i, h_a]$  define a trivial solution in GR and the local signature for metrics of type  $(+, -, -, \dots -)$ . Such metrics are written

$$\mathbf{g} = \phi^2(y^5)\eta_{\alpha\beta}du^\alpha \otimes du^\beta - (l_P)^2 \bar{h}(y^5)[dy^5 \otimes dy^5 + dy^6 \otimes dy^6 \pm dy^7 \otimes dy^7 \pm dy^8 \otimes dy^8], \quad (26)$$

where  $\eta_{\alpha\beta} = \text{diag}[1, -1, -1, \dots, 1]$  and  $\alpha, \beta, \dots = 1, 2, 3, 4$ . We shall use also generalized indices of type  ${}^1\alpha = (\alpha, 5, 6)$  and  ${}^2\alpha = ({}^1\alpha, 7, 8)$ , respectively for 6-d and 8-d models. Indices of type  ${}^2\alpha, {}^2\beta, \dots$  will run values  $1, 2, 3, 4, 5, \dots, m$ , where  $m \geq 2$ .

We consider sources for Einstein equations (23) with nonzero components defined by cosmological constant  $\Lambda$  and stress-energy tensor

$${}_{\delta}\Upsilon^\beta = \Lambda - M^{-(m+2)}\bar{K}_1(y^5), \quad {}_{\delta}\Upsilon^5_5 = {}_{\delta}\Upsilon^6_6 = \Lambda - M^{-(m+2)}\bar{K}_2(y^5), \quad (27)$$

for a fundamental mass scale  $M$  on  $TV$ ,  $\dim TV = 8$ . The fiber coordinates  $y^5, y^6, y^7, y^8$  are velocity/momentum type. Diagonal trivial Finsler brane solutions can be constructed following the methods elaborated (for extra dimensional gravity) in Refs. [41, 42, 43, 44].<sup>12</sup> A metric (26) is a solution of (23) if

$$\phi^2(y^5) = \frac{3\epsilon^2 + a(y^5)^2}{3\epsilon^2 + (y^5)^2} \text{ and } l_P \sqrt{|\bar{h}(y^5)|} = \frac{9\epsilon^4}{[3\epsilon^2 + (y^5)^2]^2}, \quad (28)$$

where  $a$  is an integration constant and the width of brane is  $\epsilon$ , with some fixed integration parameters when  $\frac{\partial^2 \phi}{\partial (y^5)^2} \big|_{y^5=\epsilon} = 0$  and  $l_P \sqrt{|\bar{h}(y^5)|} \big|_{y^5=0} = 1$ ; this states the conditions that on diagonal branes the Minkowski metric on  $TV$  is 6-d or 8-d. We get compatible (with field equations) sources (27) if

$$\begin{aligned} \bar{K}_1(y^5)M^{-(m+2)} &= \Lambda + [3\epsilon^2 + (y^5)^2]^{-2} \left[ \frac{2am(a(m+2)-3)}{3\epsilon^2} (y^5)^4 + \right. \\ &\quad \left. 2[-2a(m^2+2m+6) + 3(m+3)(1+a^2)](y^5)^2 - 6\epsilon^2 m(m-3a+2) \right], \quad (29) \\ \bar{K}_2(y^5)M^{-(m+2)} &= \Lambda + [3\epsilon^2 + (y^5)^2]^{-2} \left[ \frac{2a(m-1)(a(m+2)-4)}{3\epsilon^2} (y^5)^4 + \right. \\ &\quad \left. 4[-a(m^2+m+10) + 2(m+2)(1+a^2)](y^5)^2 - 6\epsilon^2 (m-1)(m-4a+2) \right]. \end{aligned}$$

The above formulas for  $m = 2$  are similar to those for usual 6-d diagonal brane solutions with that difference that in our case the width  $\epsilon^2 = 40M^4/3\Lambda$

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<sup>12</sup>In this paper we shall use some adapted classes of solutions from the just cited paper where the extra dimensions (2,3 etc) are analyzed in general form. Here we also note that our notations for Finsler gravity models on tangent bundles are different for those used in the above papers on 6-d, and other dimensions, brane gravity solutions.

is with extra velocity/momentum coordinates and certain constants are related to  $l_P$ . Here we also emphasize that  $y^5$  has a finite maximal value  $y_0^5$  on  $TM$  because the light velocity is finite.

The Einstein equations (23) are for the Levi-Civita connection when

$$\nabla_{2\alpha} \Upsilon^{2\alpha 2\beta} = (\sqrt{|^F \mathbf{g}|})^{-1} \mathbf{e}_{2\alpha} (\sqrt{|^F \mathbf{g}|}) \Upsilon^{2\alpha 2\beta} + \Gamma_{2\alpha 2\gamma}^{2\beta} \Upsilon^{2\alpha 2\gamma} = 0. \quad (30)$$

For our ansatz (26) and (27) with coefficients (28) and (29), such a conservation law is satisfied if

$$\frac{\partial \bar{K}_1}{\partial(y^5)} = 4(\bar{K}_2 - \bar{K}_1) \frac{\partial \ln |\phi|}{\partial(y^5)}. \quad (31)$$

We conclude that a metric (26), when the coefficients are subjected to conditions (28) – (31), defines trapping solutions containing "diagonal" extensions of GR to a 8-d  $TM$  and/or possible restrictions to 6-d and 7-d. Such solutions provide also mechanisms of corresponding gravitational trapping for fields of spins 0, 1/2, 1, 2 (see similar proofs in Refs. [41, 42, 43, 44]). The above results are in some sense expected since for diagonal configurations our model is similar to the 6-d and higher dimension ones constructed in the mentioned papers. There are two substantial differences that  $m$  is fixed to have a maximal value  $m = 4$  and that  $y^5 \leq y_0^5$  where  $y_0^5$  is determined by the maximal speed of light propagation (in the supposition that it is the same as for propagation of gravitational interactions in QG).

The behavior of physical suitable sources determined by ansatz  $\bar{K}_1(y^5)$  and/or  $\bar{K}_2(y^5)$  depends (for this class of solutions) on four parameters,  $m, \epsilon, \Lambda$  and  $a$ . This is quite surprising for QG and solutions with LV because usually it is expected that quantum effects and/or Lorenz violations may be important for distances  $\sim l_P$ . It is possible to have either  $\bar{K}_1(y^5)$  or  $\bar{K}_2(y^5)$ , or both, go to zero for corresponding choices of the mentioned four parameters. To see this we may use the analysis from the Conclusion section of Ref. [44] even on  $TM$  with finite  $y_0^5$  it is not necessary to consider  $y^5 \rightarrow \infty$ . For  $m > 4$ , which is not the case of Finsler geometry from QG dispersions, the function  $\phi^2(y^5)$  may become singular at  $y^5 \rightarrow \infty$ . Such problems can be avoided because for Finsler configurations derived from GR we can take always  $m = 1, 2, 3$  and/or consider  $y_0^5$ . We can consider that in Finsler gravity that  $\bar{K}_1(y^5) \rightarrow 0$  or  $\bar{K}_2(y^5) \rightarrow 0$  for  $y^5 \rightarrow y_0^5$ .

We do not address the question of stability of Finsler brane solutions in this work. In general, stabile configurations can be constructed for diagonal solutions which survive for nonholonomically constrained off-diagonal ones (proofs are similar to those for extra dimensional brane solutions; we shall address the problem in details in our further works).



### 3.3 Finsler brane solutions

One of the main purposes of this work is to elaborate trapping scenarios for Finsler configurations with positively nontrivial N-connections as solutions of nonholonomic gravitational equations (22). The priority of such generic off-diagonal solutions is that they allow us to distinguish the QG phenomenology and effects with LV of (pseudo) Finsler type from that described, on  $TV$  by (pseudo) Riemannian ones.

#### 3.3.1 Decoupling of equations in Einstein–Finsler gravity

We consider an ansatz (24) multiplied to  $\phi^2(y^5)$  and with non-trivial N-connection coefficients (25) and define the conditions when the coefficients generate exact solutions of (22) we get extending the solutions and sources (27). The sources are parametrized in a form similar to (A.2),

$$\begin{aligned}\tilde{\Upsilon}^\beta_\delta &= \text{diag}[\tilde{\Upsilon}^1_1 = \tilde{\Upsilon}^2_2 = \tilde{\Upsilon}_2(u^{2\alpha}), \tilde{\Upsilon}^3_3 = \tilde{\Upsilon}^4_4 = \tilde{\Upsilon}_4(u^{2\alpha}), \\ &\tilde{\Upsilon}^5_5 = \tilde{\Upsilon}^6_6 = \tilde{\Upsilon}_6(u^{2\alpha}), \tilde{\Upsilon}^7_7 = \tilde{\Upsilon}^8_8 = \tilde{\Upsilon}_8(u^{2\alpha})],\end{aligned}\quad (32)$$

when the coefficients are subjected to algebraic conditions (for vanishing N-connections, containing respectively the functions (27) determining sources in the gravitational field equations)  ${}^h\Lambda(x^i) = \tilde{\Upsilon}_4 + \tilde{\Upsilon}_6 + \tilde{\Upsilon}_8$ ,  ${}^v\Lambda(x^i, v) = \tilde{\Upsilon}_2 + \tilde{\Upsilon}_6 + \tilde{\Upsilon}_8$ ,  ${}^5\Lambda(x^i, y^5) = \tilde{\Upsilon}_2 + \tilde{\Upsilon}_4 + \tilde{\Upsilon}_8$ ,  ${}^7\Lambda(x^i, y^5, y^7) = \tilde{\Upsilon}_2 + \tilde{\Upsilon}_4 + \tilde{\Upsilon}_6$ .

Using the above assumptions on metric ansatz and sources, the conditions of Theorem A.1 can be extended step by step for dimensions 2+2+2+2. We obtain a system of equations with decoupling (separation) of partial differential equations (generalizing respectively (A.3) and (A.6)):

$$\tilde{R}^1_1 = \tilde{R}^2_2 = \frac{1}{2g_1g_2}\left[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1''\right] = -{}^h\Lambda(x^i), \quad (33)$$

$$\tilde{R}^3_3 = \tilde{R}^4_4 = \frac{1}{2h_3h_4}\left[-h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^* h_4^*}{2h_3}\right] = -{}^v\Lambda(x^i, v), \quad (34)$$

$$\tilde{R}^5_5 = \tilde{R}^6_6 = \frac{1}{2h_5h_6}\left[-\partial_{y^5y^5}^2 h_6 + \frac{(\partial_{y^5} h_6)^2}{2h_6} + \frac{(\partial_{y^5} h_5)(\partial_{y^5} h_6)}{2h_5}\right] = -{}^5\Lambda(x^i, y^5),$$

$$\tilde{R}^7_7 = \tilde{R}^8_8 = \frac{1}{2h_7h_8}\left[-\partial_{y^7y^7}^2 h_8 + \frac{(\partial_{y^7} h_8)^2}{2h_8} + \frac{(\partial_{y^7} h_7)(\partial_{y^7} h_8)}{2h_7}\right] = -{}^7\Lambda(x^i, y^5, y^7),$$

with partial derivatives on velocity/momentum type coordinates taken on respective fibers, for instance,  $\partial_{y^5} h_6 = \partial h_6 / \partial y^5$ . The equations (33) are completely similar to (A.3) and the equations (34) reproduce three times (correspondingly, for couples of variables  $(y^3 = v, y^4)$ ,  $(y^5, y^6)$ ,  $(y^7, y^8)$ ), and

"anisotropic" coordinates  $v, y^5, y^7$  and Killing symmetries on vectors  $\partial/\partial y^4$ ,  $\partial/\partial y^6$  and  $\partial/\partial y^8$ ) the equations (A.4). The equations

$$\begin{aligned}\tilde{R}_{3j} &= \frac{h_3^*}{2h_3}w_j^* + A^*w_j + B_j = 0, \\ \tilde{R}_{5j} &= \frac{\partial_{y^5}h_5}{2h_5}\partial_{y^5}{}^1w_j + (\partial_{y^5}{}^1A) {}^1w_j + {}^1B_j = 0, \\ \tilde{R}_{7j} &= \frac{\partial_{y^7}h_7}{2h_7}\partial_{y^7}{}^2w_j + (\partial_{y^7}{}^2A) {}^2w_j + {}^2B_j = 0,\end{aligned}\tag{35}$$

generalize on 8-d  $TM$  the equations (A.5). The system

$$\begin{aligned}\tilde{R}_{4i} &= -\frac{h_4^*}{2h_3}n_i^* + \frac{h_4^*}{2}K_i = 0, \quad \tilde{R}_{6i} = -\frac{\partial_{y^5}h_6}{2h_5}\partial_{y^5}{}^1n_i + \frac{\partial_{y^5}h_6}{2}{}^1K_i = 0, \\ \tilde{R}_{8i} &= -\frac{\partial_{y^7}h_8}{2h_7}\partial_{y^7}{}^2n_i + \frac{\partial_{y^7}h_8}{2}{}^2K_i = 0,\end{aligned}\tag{36}$$

is an extension of (A.6).

In the above formulas (35) and (36), there are considered nontrivial N-connection coefficients (25) and extensions of (A.7),

$$\begin{aligned}A &= (\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}), \quad B_k = \frac{h_4^*}{2h_4}(\frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2}) - \partial_k A, \quad K_1 = -\frac{1}{2}(\frac{g_1'}{g_2 h_3} + \frac{g_2^\bullet}{g_2 h_4}), \\ K_2 &= \frac{1}{2}(\frac{g_2^\bullet}{g_1 h_3} - \frac{g_2'}{g_2 h_4}); \quad {}^1A = (\frac{\partial_{y^5}h_5}{2h_5} + \frac{\partial_{y^5}h_6}{2h_6}), \quad {}^1B_k = \frac{\partial_{y^5}h_6}{2h_6}(\frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2}) \\ &- \partial_k {}^1A, \quad {}^1K_1 = -\frac{1}{2}(\frac{g_1'}{g_2 h_5} + \frac{g_2^\bullet}{g_2 h_6}), \quad {}^1K_2 = \frac{1}{2}(\frac{g_2^\bullet}{g_1 h_5} - \frac{g_2'}{g_2 h_6}); \\ {}^2A &= (\frac{\partial_{y^7}h_7}{2h_7} + \frac{\partial_{y^7}h_8}{2h_8}), \quad {}^2B_k = \frac{\partial_{y^7}h_8}{2h_8}(\frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2}) - \partial_k {}^2A, \\ {}^2K_1 &= -\frac{1}{2}(\frac{g_1'}{g_2 h_7} + \frac{g_2^\bullet}{g_2 h_8}), \quad {}^2K_2 = \frac{1}{2}(\frac{g_2^\bullet}{g_1 h_7} - \frac{g_2'}{g_2 h_8}).\end{aligned}$$

### 3.3.2 Integration of equations

The conditions of Theorem B.1 can be extended on 8-d  $TM$  which allows us to integrate in general forms the system of gravitational field equations (see respectively the equations (33), (34), (35) and (36) in EFG. Such solutions can be parametrized additionally to the data (A.10)–(A.14) (for  $g_i(x^k), h_a(x^k, v), w_i(x^k, v)$  and  $n_i(x^k, v)$ ) by coefficients

$$\begin{aligned}h_5(x^i, y^5) &= \epsilon_5 {}^0_1 h(x^i) [\partial_{y^5}{}^1 f(x^i, y^5)]^2 | {}^1\varsigma(x^i, y^5)|, \quad {}^1\varsigma = {}^0_1 \varsigma(x^i) - \frac{\epsilon_5}{8} {}^0_1 \\ h(x^i) &= \int (dy^5) {}^5\Lambda(x^i, y^5) [\partial_{y^5}{}^1 f(x^i, y^5)] [{}^1 f(x^i, y^5) - {}^0_1 f(x^i)], \\ h_6(x^i, y^5) &= \epsilon_6 [{}^1 f(x^i, y^5) - {}^0_1 f(x^i)]^2;\end{aligned}$$

$$\begin{aligned}
{}^1w_j(x^i, y^5) &= {}^1_0w_j(x^i) \exp \left\{ - \int_0^{y^5} \left[ \frac{2h_5 \partial_{y^5} ({}^1A)}{\partial_{y^5} h_5} \right]_{y^5 \rightarrow v_1} dv_1 \right\} \\
\int_0^{y^5} dv_1 &\left[ \frac{h_5 {}^1B_j}{\partial_{y^5} h_5} \right]_{y^5 \rightarrow v_1} \exp \left\{ - \int_0^{v_1} \left[ \frac{2h_5 \partial_{y^5} {}^1A}{\partial_{y^5} h_5} \right]_{y^5 \rightarrow v_1} dv_1 \right\}, \\
{}^1n_j(x^i, y^5) &= {}^1_0n_j(x^k) + \int dy^5 h_5 {}^1K_j, \\
h_7(x^i, y^5, y^7) &= \epsilon_7 {}^0_2h(x^i) [\partial_{y^7} {}^2f(x^i, y^5, y^7)]^2 |{}^2\varsigma(x^i, y^5, y^7)|, {}^2\varsigma = {}^0_2\varsigma(x^i) \\
&- \frac{\epsilon_7}{8} {}^0_2h(x^i) \int (dy^7) {}^7\Lambda [\partial_{y^7} {}^2f(x^i, y^5, y^7)] [{}^2f(x^i, y^5, y^7) - {}^0_2f(x^i)], \\
h_8(x^i, y^5, y^7) &= \epsilon_8 [{}^2f(x^i, y^5, y^7) - {}^0_2f(x^i)]^2; \\
{}^2w_j(x^i, y^5, y^7) &= {}^2_0w_j(x^i) \exp \left\{ - \int_0^{y^7} \left[ \frac{2h_7 \partial_{y^7} ({}^2A)}{\partial_{y^7} h_7} \right]_{v \rightarrow v_1} dv_1 \right\} \int_0^{y^7} dv_1 \\
&\left[ \frac{h_7 {}^2B_j}{\partial_{y^7} h_7} \right]_{y^7 \rightarrow v_1} \exp \left\{ - \int_0^{v_1} \left[ \frac{2h_7 \partial_{y^7} {}^2A}{\partial_{y^7} h_7} \right]_{y^7 \rightarrow v_1} dv_1 \right\}, \\
{}^2n_j(x^i, y^7) &= {}^2_0n_j(x^k) + \int dy^7 h_7 {}^2K_j
\end{aligned} \tag{37}$$

Such solutions with nonzero  $h_3^*, h_4^*, \partial_{y^5} h_5, \partial_{y^5} h_6, \partial_{y^7} h_7, \partial_{y^7} h_8$  are determined by generating functions  $f(x^i, v), f^* \neq 0, {}^1f(x^i, y^5), \partial_{y^5} {}^1f \neq 0, {}^2f(x^i, y^5, y^7), \partial_{y^7} {}^2f \neq 0$ , and integration functions  ${}^0f(x^i), {}^0h(x^i), {}^0w_j(x^i), {}^0n_i(x^k), {}^0_1f(x^i), {}^0_1h(x^i), {}^0_1w_j(x^i), {}^0_1n_i(x^k), {}^0_2f(x^i), {}^0_2h(x^i), {}^0_2w_j(x^i), {}^0_2n_i(x^k)$ . We should chose and/or fix such functions following additional assumptions on symmetry of solutions, boundary conditions etc.

There are substantial differences between branes in Finsler gravity and in extra dimension theories. In the first case, the physical constants/ parameters are induced in quasi-classical limits from QG on (co) tangent bundles but in the second case the constructions are for high dimensional space-time models. An important problem to be solved for such geometries is to show that there are trapping mechanisms for nonholonomic configurations to Finsler branes with finite widths (determined by the maximal value of light velocity) and possible warping on "fiber" coordinates.

### 3.3.3 On (non) diagonal brane solutions on $TM$

It is not clear what physical interpretation may have the above general solutions for Finsler gravity. We have to impose additional restrictions on some coefficients of metrics and sources in order to construct in explicit form certain Finsler brane configurations and model a trapping mechanism with generic off-diagonal metrics.

Let us consider a class of sources in EFG when for trivial N-connection coefficients (i.e. for zero values (25)) the sources  $\tilde{\Upsilon}^{2\beta}_{2\delta}$  (32) transform into data  ${}_{,1}\Upsilon^{2\beta}_{2\delta}$  (27), with nontrivial limits for  ${}_{,1}\Upsilon^\beta_\delta = \Lambda - M^{-(m+2)}\overline{K}_1(y^5)$  and  ${}_{,1}\Upsilon^5_5 = {}_{,1}\Upsilon^6_6 = \Lambda - M^{-(m+2)}\overline{K}_2(y^5)$ . The generating  $f$ -functions are taken in the form when  $h_5 = l_P \frac{\bar{h}(y^5)}{\phi^2(y^5)} {}^qh_5(x^i, y^5)$ ,  $h_6 = l_P \frac{\bar{h}(y^5)}{\phi^2(y^5)} {}^qh_6(x^i, y^5)$ ,  $h_7 = l_P \frac{\bar{h}(y^5)}{\phi^2(y^5)} {}^qh_7(x^i, y^5, y^7)$ ,  $h_8 = l_P \frac{\bar{h}(y^5)}{\phi^2(y^5)} {}^qh_8(x^i, y^5, y^7)$ , where the generating functions are parametrized in such a form that  $\phi^2(y^5)$  and  $h_5(y^5)$  are those for diagonal metrics, i.e. of type (28), and  ${}^qh_5, {}^qh_6, {}^qh_7, {}^qh_8$  are computed following formulas (A.10)–(A.14) and (37). The resulting off-diagonal solutions are

$$\mathbf{g} = g_1 dx^1 \otimes dx^1 + g_2 dx^2 \otimes dx^2 + h_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4 \mathbf{e}^4 \otimes \mathbf{e}^4 + \quad (38)$$

$$(l_P)^2 \frac{\bar{h}}{\phi^2} [{}^qh_5 \mathbf{e}^5 \otimes \mathbf{e}^5 + {}^qh_6 \mathbf{e}^6 \otimes \mathbf{e}^6 + {}^qh_7 \mathbf{e}^7 \otimes \mathbf{e}^7 + {}^qh_8 \mathbf{e}^8 \otimes \mathbf{e}^8],$$

$$\mathbf{e}^3 = dy^3 + w_i dx^i, \mathbf{e}^4 = dy^4 + n_i dx^i, \mathbf{e}^5 = dy^5 + {}^1w_i dx^i, \quad (39)$$

$$\mathbf{e}^6 = dy^6 + {}^1n_i dx^i, \mathbf{e}^7 = dy^7 + {}^2w_i dx^i, \mathbf{e}^8 = dy^8 + {}^2n_i dx^i.$$

Any solution of type (38) describes an off-diagonal trapping for 8-d (respectively, for corresponding classes of generating and integration functions, 5-, 6-, 7-d) to 4-d modifications of GR with some corrections depending on QG "fluctuations" and LV effects. There is a class of sources when for vanishing N-connection coefficients (39) we get diagonal metrics of type (24) but multiplied to a conformal factor  $\phi^2(y^5)$  when the  $h$ -coefficients are solutions of equations of type (34). Even for some diagonal limits, such metrics are very different and can not be transformed, in general form, from one to another even asymptotically, when  $\phi(y^5) \rightarrow a$  for  $y^5 \rightarrow \infty$ , they may mimic some similar behavior and QG contributions.

With respect to a local coordinate cobase  $du^{2\alpha} = (dx^i, dy^a, dy^{1a}, dy^{2a})$ , a solution (38) is parametrized by an off-diagonal matrix  $g_{2\alpha 2\beta} =$

$$\begin{bmatrix} A_{11} & A_{12} & w_1 h_3 & n_1 h_4 + & {}^1w_1 h_5 & {}^1n_1 h_6 + & {}^2w_1 h_7 & {}^2n_1 h_8 \\ A_{21} & A_{22} & w_2 h_3 & n_2 h_4 & {}^1w_2 h_5 & {}^1n_2 h_6 & {}^2w_2 h_7 & {}^2n_2 h_8 \\ w_1 h_3 & w_2 h_3 & h_3 & 0 & 0 & 0 & 0 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 & 0 & 0 & 0 & 0 \\ {}^1w_1 h_5 & {}^1w_2 h_5 & 0 & 0 & h_5 & 0 & 0 & 0 \\ {}^1n_1 h_6 & {}^1n_2 h_6 & 0 & 0 & 0 & h_6 & 0 & 0 \\ {}^2w_1 h_7 & {}^2w_2 h_7 & 0 & 0 & 0 & 0 & h_7 & 0 \\ {}^2n_1 h_8 & {}^2n_2 h_8 & 0 & 0 & 0 & 0 & 0 & h_8 \end{bmatrix}$$

where the possible observable QG and LV contributions (fluctuations in

general form) are distinguished by terms proportional to  $(l_P)^2$  in

$$\begin{aligned}
A_{11} &= g_1 + w_1^2 h_3 + n_1^2 h_4 + (l_P)^2 \frac{\bar{h}}{\phi^2} \times \\
&\quad [({}^1w_1)^2 {}^q h_5 + ({}^1n_1)^2 {}^q h_6 + ({}^2w_1)^2 {}^q h_7 + ({}^2n_1)^2 {}^q h_8], \\
A_{12} &= A_{21} = w_1 w_2 h_3 + n_1 n_2 h_4 + (l_P)^2 \frac{\bar{h}}{\phi^2} \times \\
&\quad [{}^1w_1 {}^1w_2 {}^q h_5 + {}^1n_1 {}^1n_2 {}^q h_6 + {}^2w_1 {}^2w_2 {}^q h_7 + {}^2n_1 {}^2n_2 {}^q h_8], \\
A_{22} &= g_2 + w_2^2 h_3 + n_2^2 h_4 + (l_P)^2 \frac{\bar{h}}{\phi^2} \times \\
&\quad [({}^1w_2)^2 {}^q h_5 + ({}^1n_2)^2 {}^q h_6 + ({}^2w_2)^2 {}^q h_7 + ({}^2n_2)^2 {}^q h_8].
\end{aligned}$$

It is possible to distinguish experimentally such off-diagonal metrics in Finsler geometry from diagonal configurations (26) with Levi-Civita connection on  $TM$ .

On Finsler branes determined by data (37), the gravitons are allowed to propagate in the bulk of a Finsler spacetime with dependence on velocity/momentum coordinates. The reason to introduce warped Finsler geometries and consider various trapping mechanisms is that following modern experimental data there are not explicit observations for Finsler like metrics in gravity even such dependencies can be always derived in various QG models. There are expectations that brane trapping effects may allow us to detect QG and LV effects experimentally even at scales much large than the Planck one and for different scenarios than those considered in Refs. [1, 3, 7, 5, 6, 10, 11, 12].

It is not surprising that two classes of solutions, of type (38) and (24), are very different on structure and physical implications because such metrics were subjected to the conditions to solve two different classes of gravitational field equations, respectively, (22) and (23). It should also emphasized here that conservation laws of type (30) are not satisfied for Finsler type solutions even the conditions (31) can be imposed for some initial data for  $\overline{K}_1, \overline{K}_2$  and  $\phi$ . In EFG with the Cartan d-connection, the conservation law  $\nabla_{2\alpha} \Upsilon^{2\alpha 2\beta} = 0$  is nonholonomically deformed into

$$(\sqrt{|{}^F\mathbf{g}|})^{-1} \mathbf{e}_{2\alpha} (\sqrt{|{}^F\mathbf{g}|} \Upsilon^{2\alpha 2\beta}) + \Gamma_{2\alpha 2\gamma}^{2\beta} \Upsilon^{2\alpha 2\gamma} = \tilde{\mathbf{Z}}_{2\alpha 2\gamma}^{2\beta} \Upsilon^{2\alpha 2\gamma}, \quad (40)$$

where the distortion term  $\tilde{\mathbf{Z}}_{2\alpha 2\gamma}^{2\beta}$  (21) is determined by nontrivial torsion components (19) (in their turn completely defined by generic off-diagonal terms of  $\mathbf{g}$  and respective N-connection coefficients).

Conservation laws of type (40) are typical for systems with some degrees of freedom subjected to anholonomic constraints, see (17), which is the case of Finsler spaces (more than that, possible LV result in more sophisticated local spacetime symmetries). They are derived from generalized Bianchi equations for the normal/Cartan d-connection  $\tilde{\mathbf{D}}$  [15] (the problem of formulating conservation laws for gravity theories with local gravity is discussed in Refs. [31, 30, 17]). In our approach, with respect to N-adapted frames, we can compute constraints of type (31) with some additional terms following from (40) reflecting some arbitrariness for fixing nonholonomic distributions and frames on  $TM$  for EFG. In general, the anholonomic deformation method allows us to construct Finsler brane type solutions with generic off-diagonal QG and LV terms expressed in general form (not depending explicitly on the type of metric compatible d-connection we consider, type of fundamental Finsler function and generating/integration functions).

## 4 Discussion and Conclusions

During the last decade, Finsler like gravity models were studied because they appeared to provide possible scenarios of Lorentz symmetry violations (LV) in quantum gravity (QG), new ideas for modified gravity theories with local anisotropy and in relation to dark matter and dark energy problems in modern cosmology [3, 4, 11, 12, 18, 19, 22, 25, 27, 28, 29]. The crux of the argument that QG can be related to Finsler geometry follows from three important physical results:

1. There are fundamental uncertainty relations for quantum physics,

$$x_k p_j - p_k x_j \sim i\hbar, \quad (41)$$

where  $x_i$  are operators associated to coordinates on a manifold  $V$  and  $p_j$  are momentum variables associated to  $T^*V$ , being dual to certain "velocities"  $y_k$  on  $TV$ ;  $i^2 = -1$  and  $\hbar$  is the Planck constant.

2. The bulk of QG theories are with nonlinear dispersion relations (8) which encode certain Finsler structure of type (6).
3. The general relativity (GR) theory can be written equivalently in so-called "formal" Finsler variables which can be defined on any (pseudo) Riemannian manifold with conventional horizontal (h) and (v) vertical splitting (for instance, via non-integrable 2+2 distributions/frame decompositions) [32, 45, 31, 17].

Quantum theories are, at least in quasi-classical limits, some geometric models on (co) tangent bundles of certain manifolds endowed with geometric, dynamical and nonholonomic structure adapted to non-integrable distributions on  $TV$ , or  $T^*V$ , determined by generating functions  $F(x, y)$  (in particular, of uncertainty type (41)). The principle of equivalence in GR imposes via nonlinear dispersion relations (6) and (8) the condition that  $F(x, y)$  is a homogeneous on  $y$ -variables Finsler metric, see details in Refs. [26, 13, 11, 12, 10]. Generalizations of the principle of general covariance and axiomatics of GR to  $TV$  result in theories with arbitrary (nonhomogeneous)  $F$  and more general metric structures  $g_{\alpha\beta}(x, y)$  and frame transforms and deformations.

A number of papers on Finsler gravity and applications written by physicists are restricted only to models with "nonlinear" quadratic Finsler elements  $ds^2 = F^2(x, y)$  without important studies of physical implications of nonlinear and distinguished connection structures. Non-experts in Finsler geometry consider that locally anisotropic theories are completely defined by  $F$  in a form which is similar to (pseudo) Riemannian geometry which is completely determined by a quadratic  $({}^0F)^2 = g_{ij}(x)y^i y^j$ , for  $y^i \sim dx^i$ . Really, in GR a metric tensor field  $g_{ij}(x)$  defines a unique metric compatible Levi-Civita connection  $\nabla$  on  $V$ , and  $TV$ , and corresponding fundamental Riemann/Ricci/Einstein tensors when the torsion field is constrained to be zero. Nevertheless, this is not true for Finsler geometries and related gravity models because in such approaches the geometric constructions are based on three fundamental geometric objects: a total metric,  ${}^F\mathbf{g}$ , a nonlinear connection,  ${}^F\mathbf{N}$ , and a distinguished (adapted) linear connection,  ${}^F\mathbf{D}$ . For certain well-defined geometric/physical principles, all such values are uniquely generated by  $F$  and this means that a Finsler space<sup>13</sup> is defined by a triple of geometric data  $(F : \mathbf{g}, \mathbf{N}, \mathbf{D})$ . Finsler theories are with more rich geometric structures than the (pseudo) Riemannian ones determined by data  $(\mathbf{g}, \nabla)$ .

In order to elaborate a self-consistent geometric model of classical and quantum Finsler gravity theory we have to involve into constructions all fundamental geometric/physical objects. Such values must be included into certain gravitational and matter field gravitational field equations (derived on  $TM$  following certain generalized variational/geometric principles). It is also necessary to try to perform a quantization program and then to analyze possible consequences/applications, for instance, in modern cosmology and astrophysics, or geometric mechanics, see details on such a series of works

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<sup>13</sup>spacetime, if  ${}^F\mathbf{g}$  is related to a Minkowski metric in special relativity, or GR; for simplicity, we omit the left label  $F$  if this does not result in ambiguities

in Refs. [31, 17, 13, 33, 34, 40].

We must solve two important problems for quantum/noncommutative Finsler generalizations of GR:

- What type of Finsler nonlinear and linear connections,  $({}^F\mathbf{N}, {}^F\mathbf{D})$ , are chosen following certain geometric and physical arguments? For instance, mathematicians [16] prefer to work with the Chern and Berwald connections which are metric noncompatible (certain cosmological models [28, 29] were elaborated following such an approach). Nevertheless, constructions with metric noncompatible connections are less relevant to generalizations of standard theories of particle physics because does not allow to define in a usual form a particle classification, Dirac equations, conservation laws etc, see critical remarks in [30, 31, 17]. In our works, we preferred to elaborate physical models when  ${}^F\mathbf{D}$  is chosen to be the canonical distinguished and/or normal/Cartan distinguished connections. Such constructions are metric compatible and allow "more standard" theories of Finsler extension of GR (the so-called Einstein–Finsler gravity, EFG, models).
- Another problem is that if existing experimental data do not constrain "too much" the perspectives of Finsler gravity for realistic QG and LV theories? For instance, in Ref. [10], such an analysis is performed with the conclusion that coefficients  $q_{\hat{i}_1\hat{i}_2...\hat{i}_{2r}}$  in a Finsler metric (6) and a related dispersion relation (8) seem to be very small and this sounds to be very pessimistic for detecting a respective QG phenomenology and LV. Here we note that a conclusion drawn only using certain data for a Finsler metric  $F(x, y)$  is not a final one because any parametrization (6) is "geometric gauge" dependent. Really, using frame/coordinate transforms and nonholonomic deformations,

$$(F : \mathbf{g}, \mathbf{N}, \mathbf{D}) \rightarrow ({}^0F : \check{\mathbf{g}}, \check{\mathbf{N}}, \check{\mathbf{D}})$$

when  ${}^0F$  is a typical quadratic form in GR, the LV effects are removed into data  $(\check{\mathbf{N}}, \check{\mathbf{D}})$  modeling nonlinear generic off-diagonal quantum, and quasi-classical, interactions in QG. An explicit example of such systems is that of noncommutative Finsler black holes [26]. The black hole solutions and various gravitational–gauge–fermion interactions, in GR and EFG can not be studied experimentally only via Mikelson–Morley and possible related nonlinear dispersion effects determined only by  $F$ . Off-diagonal metrics and anholonomic frames (nonlinear connection) and induced torsion effects (distinguished connection) effects are of crucial importance in Finsler theories.



A rigorous mathematical and physically motivated approach should consider formulations of Finsler gravity theories for certain physically important classes of nonlinear and distinguished connections. We should try to find exact solutions and after that to analyze possible physical implications not restricting our approach only to  $F$  but to complete theories with nontrivial  $\mathbf{N}$  and  $\mathbf{D}$ . Surprisingly, such solutions can be constructed in very general off-diagonal forms [32, 45, 31, 17] and we apply such methods in this paper. Nontrivial Finsler spaces on  $TV$  are with generic off-diagonal metrics  $\mathbf{g}$  which in certain coordinate bases contain contributions from  $\mathbf{N}$ . Various limits from EFG to GR can be modeled by a corresponding nonholonomic and nonlinear dynamics when the coefficients of metrics depend anisotropically at least on 3–5–7 space, time and velocity type coordinates on 8-d  $TV$  Finsler spacetimes. Such scenarios are more complex than the well-known compactification of extra dimensions in Kaluza–Klein gravity. Even diagonal metrics for Finsler–Kaluza–Klein gravity can be used for certain rough estimations, we need more sophisticated classes of generic off-diagonal exact solutions with warping and trapping of interactions in order to get a constant nonzero value for the speed of light and generic off-diagonal "bulk" configurations with nontrivial  $\mathbf{N}$ .

In this article, we have constructed brane world solutions of gravitational field equations for metric compatible EFG theories of QG and possible LV. We found that using generic off-diagonal metrics, non-integrable constraints, Finsler connections and stress-energy ansatz functions it is possible to realize trapping gravitational configurations with physically reasonable properties for a range of parameters (for instance, the extra dimension,  $m \geq 1$ ; bulk cosmological constant  $\Lambda$ , in general, with locally anisotropic polarizations; brane width  $\epsilon$ ; a constant value  $a$  of gravitational interactions for the maximal speed of light/ gravitational interactions etc). Such brane effects of QG with LV depend on the mechanism of Finsler type gravitational and matter fields interactions on tangent bundle  $TV$  over a spacetime  $V$  in general relativity (GR) and it is expected that they may be detected in TeV physics, or via modifications in modern cosmology and astrophysics (on locally anisotropic Finsler cosmological scenarios and exact solutions with black ellipsoids, wormholes etc see [40, 17]).

The solutions with trapping from  $TV$  to a GR spacetime  $V$  are of two general forms: The first class consists from almost standard diagonal generalizations/ modifications of results from Refs. [41, 42] for 6-d (and higher) dimensions, when the Einstein equations for the Levi–Civita connection were extended to 8-d (pseudo) Riemannian spacetimes, with possible two time coordinates. If such diagonal brane effects of QG and/or LV origin can be

detected experimentally, we can conclude that QG gravity is a (co) tangent bundle geometric theory for the Levi–Civita connection determined by special types of nonlinear dispersions (generating a trivial nonlinear connection, N–connection structure).

Nevertheless, very general assumptions on LV effects and nonlinear dispersions induced from QG seem to result in a second class of Finsler like nonlinear quadratic elements and canonically induced linear connections (for instance, the so–called normal/Cartan d–connection) which are different from the well known Levi–Civita connection. On  $TV$ , it is naturally to work with metric compatible d–connections with effective torsion completely determined by a (Finsler) metric and N–connection structure as we discussed in details in Refs. [31, 30, 13, 17]. To construct brane solutions with nontrivial N–connection structure and generic off–diagonal metrics (the first attempts where considered in [40, 17]) is a more difficult technical task which can be solved following the so–called anholonomic deformation/frame method [32, 45]. We shall address possible applications of nonholonomic geometry methods and Finsler brane solutions in (non) commutative locally anisotropic cosmology and black holes physics [26, 13, 33].

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## A Einstein–Finsler Spaces of Dimension 2+2

In this Appendix, we study a toy model of Einstein–Finsler gravity on  $TM$  over a 2–dimensional manifold  $M$ . We prove that such a theory can be integrated in general form. Local coordinates are labeled  $u^\alpha = (x^k, y^a)$ , where indices run respectively the values:  $i, j, k, \dots = 1, 2$ ;  $a, b, c, \dots = 3, 4$ ; and  $y^3 = v$ . Using frame transforms any (pseudo) Finsler/ Riemannian 4–d metric can parametrized in the form

$$\begin{aligned} \mathbf{g} &= g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, y^b)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a \\ &\text{for } \mathbf{e}^3 = dy^3 + w_i(x^k, v)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i, \end{aligned} \quad (\text{A.1})$$

which is a particular case of (14). We label in brief the partial derivatives in the form  $g_1^\bullet = \partial g_1 / \partial x^1$ ,  $g_1' = \partial g_1 / \partial x^2$  and  $h_3^* = \partial h_3 / \partial v$ .

For Finsler configurations, the condition of homogeneity results in at least on Killing symmetry for metrics. We can always introduce such a N–adapted frame/coordinate parametrization when  $\omega^2 = 1$  and the above metric does not depend on variables  $y^4$ . The coefficients of the normal/ Cartan d–connection  $\tilde{\Gamma}_{\alpha\beta}^\gamma$  (20) can be computed for a metric (A.1) with  $\omega^2 = 1$ ,

when  $g_{\alpha\beta} = \text{diag}[g_i(x^k), h_a(x^i, v)]$  and  $N_k^3 = w_k(x^i, v)$ ,  $N_k^4 = n_k(x^i, v)$ .<sup>14</sup> Using the Cartan structure equations (18), it is possible to determine the  $h$ - and  $v$ -components of the Riemannian, torsion, Ricci d-tensors etc.

For the 2+2 dimensional EFG theory, there is a very important property of decoupling/separation of field equations with respect to a class of N-adapted frames which allows us to integrate the theory in very general forms (see Theorem B.1) depending on the types of prescribed nonholonomic constraints and given sources parametrized by frame transform as

$$\tilde{\Upsilon}_\beta^\delta = \text{diag}[\tilde{\Upsilon}_1^1 = \tilde{\Upsilon}_2^2 = {}^v\Lambda(x^i, v), \tilde{\Upsilon}_3^3 = \tilde{\Upsilon}_4^4 = {}^h\Lambda(x^i)]. \quad (\text{A.2})$$

As particular cases, such sources generalize contributions from nontrivial cosmological constants (for instance, if  ${}^h\Lambda = {}^v\Lambda = \Lambda = \text{const}$ ), their non-holonomic matrix polarizations, approximations for certain dust/radiation locally anisotropic states of matter etc.

**Theorem A.1** *The Finsler gravitational field equations (22) for a metric (A.1) with  $\omega^2 = 1$  and source (A.2) are equivalent to this system of partial differential equations:*

$$\tilde{R}_1^1 = \tilde{R}_2^2 = \frac{1}{2g_1g_2} \left[ \frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' \right] = -{}^h\Lambda \quad (\text{A.3})$$

$$\tilde{R}_3^3 = \tilde{R}_4^4 = \frac{1}{2h_3h_4} \left[ -h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^* h_4^*}{2h_3} \right] = -{}^v\Lambda, \quad (\text{A.4})$$

$$\tilde{R}_{3j} = \frac{h_3^*}{2h_3} w_j^* + A^* w_j + B_j = 0, \quad (\text{A.5})$$

$$\tilde{R}_{4i} = -\frac{h_4^*}{2h_3} n_i^* + \frac{h_4^*}{2} K_i = 0, \quad (\text{A.6})$$

$$A = \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right), \quad B_k = \frac{h_4^*}{2h_4} \left( \frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2} \right) - \partial_k A, \quad (\text{A.7})$$

$$K_1 = -\frac{1}{2} \left( \frac{g_1'}{g_2 h_3} + \frac{g_2^\bullet}{g_2 h_4} \right), \quad K_2 = \frac{1}{2} \left( \frac{g_2^\bullet}{g_1 h_3} - \frac{g_2'}{g_2 h_4} \right).$$

**Proof.** We apply the constructions for the canonical d-connection from [32, 45, 17] to the case of normal/ Cartan d-connection on 4-d  $TM$ . Following definition of coefficients  $\tilde{\Upsilon}_\alpha^\gamma{}_\beta$  (20), the  $h$ - and  $v$ -components are similar to those for those for the canonical d-connection. So, the proofs for equations (A.3) and (A.4) are completely similar to those for presented in

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<sup>14</sup>Following methods elaborated in Refs. [45, 32], we can construct exact solutions with  $\omega^2 \neq 1$ . For Finsler brane configurations, for simplicity, we do not consider such "very" general classes of solutions.

the mentioned works. For  $\tilde{\Gamma}_{\alpha\beta}^{\gamma}$ , there are differences for (A.5) and (A.6) which are analyzed in this Appendix.

We perform a N-adapted differential calculus if instead of partial derivatives  $\partial_{\alpha} = \partial/\partial u^{\alpha}$  there are considered operators (15) parametrized in the form  $\mathbf{e}_i = \partial_i - N_i^a \partial_a = \partial_i - w_i \partial_v - n_i \partial_4$ . For  $N_k^3 = w_k(x^i, v)$ ,  $N_k^4 = n_k(x^i, v)$ , the nontrivial coefficients of N-connection curvature are

$$\Omega_{12}^3 = w_2^{\bullet} - w_1' - w_1 w_2^* + w_2 w_1^*, \Omega_{12}^4 = n_2^{\bullet} - n_1' - w_1 n_2^* + w_2 n_1^*. \quad (\text{A.8})$$

There are nontrivial coefficients of  $\tilde{\Gamma}_{\alpha\beta}^{\gamma}$ ,

$$\begin{aligned} \tilde{L}_{11}^1 &= \frac{g_1^{\bullet}}{2g_1}, \tilde{L}_{12}^1 = \frac{g_1'}{2g_1}, \tilde{L}_{22}^1 = -\frac{g_2^{\bullet}}{2g_1}, \tilde{L}_{11}^2 = -\frac{g_1'}{2g_2}, \tilde{L}_{12}^2 = \frac{g_2^{\bullet}}{2g_2}, \\ \tilde{L}_{22}^2 &= \frac{g_2'}{2g_2}, \tilde{C}_{33}^3 = \frac{h_3^*}{2h_3}, \tilde{C}_{44}^3 = -\frac{h_4^*}{2h_3}, \tilde{C}_{34}^4 = \frac{h_4^*}{2h_4}. \end{aligned} \quad (\text{A.9})$$

The nontrivial coefficients of torsion (19) are

$$\begin{aligned} \tilde{T}_{12}^3 &= \Omega_{21}^3, \tilde{T}_{12}^4 = \Omega_{21}^4, \tilde{P}_{i3}^3 = w_i^* - \frac{\partial_i g_1}{2g_1}, \tilde{P}_{14}^3 = -\frac{g_1'}{2g_1}, \tilde{P}_{24}^3 = \frac{g_2^{\bullet}}{2g_1}, \\ \tilde{P}_{13}^4 &= n_1^* + \frac{g_1'}{2g_2}, \tilde{P}_{23}^4 = n_2^* - \frac{g_2^{\bullet}}{2g_1}, \tilde{P}_{i4}^4 = -\frac{\partial_i g_2}{2g_2}. \end{aligned}$$

The h-v components of the Ricci tensor are derived from

$$\begin{aligned} \tilde{R}_{bka}^c &= \frac{\partial \tilde{L}_{bk}^c}{\partial y^a} - \tilde{C}_{ba|k}^c + \tilde{C}_{bd}^c \tilde{P}_{ka}^d \\ &= \frac{\partial \tilde{L}_{bk}^c}{\partial y^a} - \left( \frac{\partial \tilde{C}_{ba}^c}{\partial x^k} + \tilde{L}_{dk}^c \tilde{C}_{ba}^d - \tilde{L}_{bk}^d \tilde{C}_{da}^c - \tilde{L}_{ak}^d \tilde{C}_{bd}^c \right) + \tilde{C}_{bd}^c \tilde{P}_{ka}^d. \end{aligned}$$

Contracting indices, we get  $\tilde{R}_{bk} = \tilde{R}_{bka}^a = \frac{\partial \tilde{L}_{bk}^a}{\partial y^a} - \tilde{C}_{b|k}^a + \tilde{C}_{bd}^a \tilde{P}_{ka}^d$ , where  $\tilde{C}_b = \tilde{C}_{ba}^c$  and  $\partial \tilde{L}_{bk}^a / \partial y^a = 0$  for (A.1) with  $\omega^2 = 1$ . We have

$$\begin{aligned} \tilde{C}_{b|k} &= \mathbf{e}_k \tilde{C}_b - \tilde{L}_{bk}^d \tilde{C}_d = \partial_k \tilde{C}_b - N_k^e \partial_e \tilde{C}_b - \tilde{L}_{bk}^d \tilde{C}_d \\ &= \partial_k \tilde{C}_b - w_k \tilde{C}_b^* - \tilde{L}_{bk}^d \tilde{C}_d, \end{aligned}$$

for  $\tilde{C}_3 = \tilde{C}_{33}^3 + \tilde{C}_{34}^4 = \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}$ ,  $\tilde{C}_4 = \tilde{C}_{43}^3 + \tilde{C}_{44}^4 = 0$ , see (A.9).

We express  $\tilde{R}_{bk} = {}^1\tilde{R}_{bk} + {}^2\tilde{R}_{bk} + {}^3\tilde{R}_{bk}$ , where

$$\begin{aligned} {}^1\tilde{R}_{bk} &= \left( \tilde{L}_{bk}^4 \right)^* = 0, {}^2\tilde{R}_{bk} = -\tilde{C}_{b|k} = -\partial_k \tilde{C}_b + w_k \tilde{C}_b^* + \tilde{L}_{bk}^d \tilde{C}_d, \\ {}^3\tilde{R}_{bk} &= \tilde{C}_{bd}^a \tilde{P}_{ka}^d = \tilde{C}_{b3}^3 \tilde{P}_{k3}^3 + \tilde{C}_{b4}^3 \tilde{P}_{k3}^4 + \tilde{C}_{b3}^4 \tilde{P}_{k4}^3 + \tilde{C}_{b4}^4 \tilde{P}_{k4}^4. \end{aligned}$$

Then, it is possible to compute  $\tilde{R}_{3k} = {}^2\tilde{R}_{3k} + {}^3\tilde{R}_{3k}$  when, for instance,  $\hat{L}_{3k}^3 \rightarrow \hat{L}_{1k}^1$  and  $\hat{L}_{4k}^4 \rightarrow \hat{L}_{2k}^2$ , with

$$\begin{aligned} {}^2\tilde{R}_{3k} &= -\partial_k \tilde{C}_3 + w_k \tilde{C}_3^* + \tilde{L}_{3k}^3 \tilde{C}_3 \\ &= -\partial_k \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right) + \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right)^* w_k + \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right) \frac{\partial_k g_1}{2g_1}, \\ {}^3\tilde{R}_{3k} &= \tilde{C}_{33}^3 \tilde{P}_{k3}^3 + \tilde{C}_{34}^3 \tilde{P}_{k3}^4 + \tilde{C}_{33}^4 \tilde{P}_{k4}^3 + \tilde{C}_{34}^4 \tilde{P}_{k4}^4 \\ &= \frac{h_3^*}{2h_3} \left( w_i^* - \frac{\partial_i g_1}{2g_1} \right) - \frac{h_4^*}{2h_4} \frac{\partial_i g_2}{2g_2}. \end{aligned}$$

(A.5) can be obtained summarizing above formulas,

$$\tilde{R}_{3k} = \frac{h_3^*}{2h_3} w_i^* + \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right)^* w_k - \partial_k \left( \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right) + \frac{h_4^*}{2h_4} \left( \frac{\partial_k g_1}{2g_1} - \frac{\partial_k g_2}{2g_2} \right).$$

Similarly, we compute  $\tilde{R}_{4k} = {}^2\tilde{R}_{4k} + {}^3\tilde{R}_{4k}$ , where

$$\begin{aligned} {}^2\tilde{R}_{4k} &= -\partial_k \tilde{C}_4 + w_k \tilde{C}_4^* + \tilde{L}_{4k}^3 \tilde{C}_4 = 0; \\ {}^3\tilde{R}_{4k} &= \tilde{C}_{43}^3 \tilde{P}_{k3}^3 + \tilde{C}_{44}^3 \tilde{P}_{k3}^4 + \tilde{C}_{43}^4 \tilde{P}_{k4}^3 + \tilde{C}_{44}^4 \tilde{P}_{k4}^4 = \tilde{C}_{44}^3 \tilde{P}_{k3}^4 + \tilde{C}_{43}^4 \tilde{P}_{k4}^3. \end{aligned}$$

Putting together, we obtain (A.6) [which ends the proof of Theorem A.1],

$$\tilde{R}_{41} = -\frac{h_4^*}{2h_3} (n_1^* + \frac{g_1'}{2g_2}) - \frac{g_2^\bullet}{2g_2} \frac{h_4^*}{2h_4}, \quad \tilde{R}_{42} = -\frac{h_4^*}{2h_3} (n_2^* - \frac{g_2^\bullet}{2g_1}) - \frac{g_2'}{2g_2} \frac{h_4^*}{2h_4}.$$

## B Integration of field equations

**Theorem B.1** *The general solutions of equations (A.3)–(A.6) defining Einstein–Finsler spaces are parametrized by metrics of type (A.1) with coefficients computed in the form*

$$g_i = \epsilon_i e^{\psi(x^k)}, \text{ for } \epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = {}^h\Lambda(x^k); \quad (\text{A.10})$$

$$h_3 = \epsilon_3 {}^0h(x^i) [f^*(x^i, v)]^2 |\varsigma(x^i, v)|, \quad (\text{A.11})$$

$$\varsigma = {}^0\varsigma(x^i) - \frac{\epsilon_3}{8} {}^0h(x^i) \int (dv) {}^v\Lambda(x^k, v) f^*(x^i, v) [f(x^i, v) - {}^0f(x^i)],$$

$$h_4 = \epsilon_4 [f(x^i, v) - {}^0f(x^i)]^2; \quad (\text{A.12})$$

$$w_j = {}^0w_j(x^i) \exp \left\{ - \int_0^v [2h_3 A^*/h_3^*]_{v \rightarrow v_1} dv_1 \right\} \times \quad (\text{A.13})$$

$$\int_0^v dv_1 [h_3 B_j/h_3^*]_{v \rightarrow v_1} \exp \left\{ - \int_0^{v_1} [2h_3 A^*/h_3^*]_{v \rightarrow v_1} dv_1 \right\},$$

$$n_i = {}^0n_i(x^k) + \int dv {}^h_3 K_i. \quad (\text{A.14})$$

Such solutions with  $h_3^*, h_4^* \neq 0$  are determined by generating,  $f(x^i, v), f^* \neq 0$ , and integration,  ${}^0f(x^i), {}^0h(x^i), {}^0w_j(x^i), {}^0n_i(x^k)$ , functions.

**Proof.** We sketch a proof following two steps:

1. **Solutions with Killing symmetry for h- and v-components of metric:** The equation (A.3) is for a two dimensional (semi) Riemannian metric. Any such metric can be diagonalized and expressed as a conformally flat metric. Choosing  $\epsilon_i e^{\psi(x^k)}$ , we get the Poisson equation in (A.10). The equation (A.4) is similar to that for the canonical d-connection configurations which was solved in general form [32, 45, 17]. Such equations relate two unknown functions. For instance, if we prescribe any  $h_3(x^i, v)$ , we can construct (at least via some series decompositions)  $h_4(x^i, v)$ , and inversely. By straightforward computations, we can verify that any  $h_3$  and  $h_4$  with nonzero  $h_3^*$  and  $h_4^*$  given by (A.11) define exact solutions for (A.4). Solutions with  $h_3^* = 0$  and/or  $h_4^* = 0$  should be re-considered as some particular degenerated cases.
2. **Solutions for the N-connection coefficients:** The main differences between our former results for the canonical d-connection and the normal/ Cartan d-connection (in this work) consist in equations (A.5) and (A.6) and coefficients (A.7). We provide the proofs of formulas (A.12) and (A.14)) in Appendix B. Taking together the solutions (A.10)–(A.14) for ansatz (A.1) with  $\omega^2 = 1$ , we constrict the general class of exact solutions with Killing symmetry on  $\partial/\partial y^4$  defining Einstein-Finsler spaces<sup>15</sup>. Considering different types of frame transforms, with coordinates parametrized for tangent bundles, such metrics can be transformed into standard ones  ${}^F\mathbf{g}$  (10) Finsler spaces.

#### Some computations for Theorem B.1:

The solutions of (A.5) and (A.6) can be always considered for  $|g_1| = |g_2|$ , when  $B_k = \partial_k A$ . We construct them for three more special cases.

Case 1:  $h_3^* = 0, h_4^* \neq 0$  and  $A = h_4^*/2h_4$ . We must solve the equation  $h_4^{**} - \frac{(h_4^*)^2}{2h_4} = 2h_3h_4 {}^v\Lambda(x^i, v)$ , for any given  $h_3 = h_3(x^i)$  and  ${}^v\Lambda(x^i, v)$ . We have  $w_j^* = 0$  and we obtain, from (A.5),  $w_j = -B_j/A^* = -\partial_j A/A^*$  and, from (A.6),  $n_i^* = K_i h_3$ .

Case 2:  $h_4^* = 0$ , any  $h_3$  and  $n_i$  for  ${}^v\Lambda = 0$ . Let us consider in (A.5) that  $h_3 \neq 0$ . We have to solve  $\frac{h_3^*}{2h_3} w_j^* + A^* w_j + B_j = 0$ . Representing  $w_j = {}^1w_j \cdot {}^2w_j$

<sup>15</sup>such a symmetry exists if the coefficients of metrics do not depend on coordinate  $y^4$

and introducing  $w_j^* = {}^1w_j^* \cdot {}^2w_j + {}^1w_j \cdot {}^2w_j^*$  into above equation, we obtain

$${}^1w_j^* \cdot {}^2w_j + {}^1w_j \cdot {}^2w_j^* + \frac{2h_3A^*}{h_3^*} {}^1w_j \cdot {}^2w_j + \frac{2h_3B_j}{h_3^*} = 0.$$

We can chose  ${}^1w_j = -{}^1_0w_j(x^i) \exp \left[ - \int \frac{2h_3A^*}{h_3^*} dv \right]$ , for some integration functions  ${}^1_0w_j(x^i)$ , and transform the equation into  ${}^2w_j^* = 2 {}^1_0w_j(x^i) \int_{v_2}^v dv_1 \frac{h_3B_j}{h_3^*} \exp \left[ - \int_{v_0}^{v_1} \frac{2h_3A^*}{h_3^*} dv_1 \right]$ , which can be integrated in general form. Finally, we get

$$w_j = {}^1_0w_j(x^i) \exp \left[ - \int \frac{2h_3A^*}{h_3^*} dv \right] \int_{v_2}^v dv_1 \frac{h_3B_j}{h_3^*} \exp \left[ - \int_{v_0}^{v_1} \frac{2h_3A^*}{h_3^*} dv_1 \right],$$

for some  $v_0, v_2 = \text{const.}$  The solution of (A.6) is constructed by a direct integration on  $v$  of values  $K_i$  from (A.7).

Case 3:  $h_3^* \neq 0, h_4^* \neq 0$ , which is stated in Theorem B.1. As a general solution of (A.5), we can consider (A.12) with the coefficients  $A, B_j$  and  $K_i$  and computed for arbitrary  $h_3$  and  $h_4$  depending on  $v$ . Integrating (A.6), we get the formula (A.14).

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